ME451: Control Systems

Lecture 7
Linearization, time delays

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems
  - Electrical
  - Mechanical
  - Electromechanical
- Block diagrams
- Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)
What is a linear system?

- A system having **Principle of Superposition**

\[
\begin{align*}
u(t) & \rightarrow y(t) \\
u_1(t) \rightarrow y_1(t) & \\
u_2(t) \rightarrow y_2(t)
\end{align*}
\]

\[
\Rightarrow \alpha_1 u_1(t) + \alpha_2 u_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t) \\
\forall \alpha_1, \alpha_2 \in \mathbb{R}
\]

A nonlinear system does not satisfy the principle of superposition.

Linear systems

- Easier to understand and obtain solutions
- Linear ordinary differential equations (ODEs),
  - Homogeneous solution and particular solution
  - Transient solution and steady state solution
  - Solution caused by initial values, and forced solution
- Add many simple solutions to get more complex ones (use superposition!)
- Easy to check the Stability of stationary states (Laplace Transform)
Why linearization?

- Real systems are inherently nonlinear. (Linear systems do not exist!) Ex. \( f(t) = Kx(t), v(t) = Ri(t) \)
- TF models are only for linear time-invariant (LTI) systems.
- Many control analysis/design techniques are available for linear systems.
- Nonlinear systems are difficult to deal with mathematically.
- Often we linearize nonlinear systems before analysis and design. How?

How to linearize it?

- Nonlinearity can be approximated by a linear function for small deviations \( \delta x \) around an operating point \( x_0 \)
- Use a Taylor series expansion

\[
 f(x_0 + \delta x) 
\]
Linearization

- Nonlinear system: \( \dot{x} = f(x, u) \)
- Let \( u_0 \) be a nominal input and let the resultant state be \( x_0 \)
- Perturbation: \( u(\cdot) = u_0(\cdot) + \delta u(\cdot) \)
- Resultant perturb: \( x(\cdot) = x_0(\cdot) + \delta x(\cdot) \)
- Taylor series expansion:
  \[
  f(x, u) = f(x_0, u_0) + \frac{\partial f(x, u)}{\partial x} |_{x=x_0, u=u_0} \delta x + \frac{\partial f(x, u)}{\partial u} |_{x=x_0, u=u_0} \delta u + \mathbf{H} \cdot \mathbf{Q} \cdot \mathbf{T} \approx 0
  \]

\[
\dot{x}_0 + \delta \dot{x} = f(x_0, u_0) + \frac{\partial f(x, u)}{\partial x} |_{x=x_0, u=u_0} \delta x + \frac{\partial f(x, u)}{\partial u} |_{x=x_0, u=u_0} \delta u
\]

notice that \( \dot{x}_0 = f(x_0, u_0) \); hence

\[
\delta \dot{x} = \frac{\partial f(x, u)}{\partial x} |_{x=x_0, u=u_0} \delta x + \frac{\partial f(x, u)}{\partial u} |_{x=x_0, u=u_0} \delta u
\]
Linearization of a pendulum model

- **Motion of the pendulum**

\[ mL^2 \ddot{\theta}(t) + mgL \sin \theta(t) = u(t) \]
\[ \ddot{\theta}(t) + \frac{g \sin \theta(t)}{L} - \frac{u(t)}{mL^2} = 0 \]

- **Linearize it at** \( \theta_0 = \pi \)
- **Find** \( u_0 \)
  \[ \ddot{\theta} + \frac{g \sin \pi}{L} - \frac{u_0}{mL^2} = 0 \rightarrow u_0 = 0 \]
- **New coordinates:**
  \[ \theta = \theta_0 + \delta \theta = \pi + \delta \theta \]
  \[ u = u_0 + \delta u = 0 + \delta u \]

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Linearization of a pendulum model (cont’)

- **Taylor series expansion** of \( f(\theta, u) \) at \( \theta = \pi, u = 0 \)

\[ \frac{\partial f(\theta, u)}{\partial \theta} \bigg|_{\theta=\pi, u=0} = \frac{g \cos \theta}{L} \bigg|_{\theta=\pi} = -\frac{g}{L} \]
\[ \frac{\partial f(\theta, u)}{\partial u} \bigg|_{\theta=\pi, u=0} = -\frac{1}{mL^2} \]

\[ \delta \ddot{\theta} + \frac{\partial f(\theta, u)}{\partial \theta} \bigg|_{\theta=\pi, u=0} \delta \theta + \frac{\partial f(\theta, u)}{\partial u} \bigg|_{\theta=\pi, u=0} \delta u = 0 \]

\[ \delta \ddot{\theta} - \frac{g}{L} \delta \theta - \frac{1}{mL^2} \delta u = 0 \]
Time delay transfer function

- TF derivation
  \[ y(t) = x(t - T_d) \quad (T_d: \text{ delay time}) \]

  \[ \mathcal{L} \quad Y(s) = e^{-T_d s} X(s) \quad \frac{Y(s)}{X(s)} = e^{-T_d s} \]

  (Memorize this!)

- The more time delay is, the more difficult to control (Imagine that you are controlling the temperature of your shower with a very long hose. You will either get burned or frozen!)

Summary and Exercises

- Modeling of
  - Nonlinear systems
  - Systems with time delay

- Next
  - Modeling of DC motors

- Exercises
  - Linearize the pendulum model at \( \pi/4 \)