Course roadmap

**Modeling**
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

**Analysis**
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

**Design**
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)
Time-invariant & time-varying

- A system is called **time-invariant (time-varying)** if system parameters do not (do) change in time.
- Example: \( Mx''(t) = f(t) \) & \( M(t)x''(t) = f(t) \)
- For time-invariant systems:

  ![Time shift diagram]

  \[ u(t) \quad \text{Time shift} \quad y(t) \]
  \[ t_0 \quad t_0 + T \quad t_0 \quad t_0 + T \]

- This course deals with time-invariant systems.

Newton’s laws of motion

- 1\(^{\text{st}}\) law:
  - A particle remains at rest or continues to move in a straight line with a constant velocity if there is no unbalancing force acting on it.

- 2\(^{\text{nd}}\) law:
  - \( \sum F_i(t) = m \frac{d^2x}{dt^2} : \) translational
  - \( \sum \tau_i(t) = I \frac{d^2\theta}{dt^2} : \) rotational

- 3\(^{\text{rd}}\) law:
  - For every action has an equal and opposite reaction
Translational mechanical elements: (constitutive equations)

Mass

\[ f(t) = M x''(t) \]
\[ (x(0) = 0) \]

Spring

\[ f(t) = K (x_1(t) - x_2(t)) \]
\[ (x_1(0) = 0, x_2(0) = 0) \]

Damper

\[ f(t) = B (x'_1(t) - x'_2(t)) \]

\[ F(s) = M s^2 X(s) \]
\[ F(s) = K (X_1(s) - X_2(s)) \]
\[ F(s) = B s (X_1(s) - X_2(s)) \]

Mass-spring-damper system

\[ M x''(t) + B x'(t) + K x(t) = f(t) \]
Free body diagram

\[ f_k(t) \quad f_b(t) \]

Direction of actual force will be automatically determined by the relative values!

\[ f_k(t) = K(x(t) - 0) \quad f_b(t) = B(x'(t) - 0) \]

- Newton’s law: \( F=ma \)

\[ M x''(t) = f(t) - f_k(t) - f_b(t) = f(t) - Kx(t) - Bx'(t) \]

Mass-spring-damper system

- Equation of motion

\[ M x''(t) + Bx'(t) + Kx(t) = f(t) \]

- By Laplace transform (with zero initial conditions),

\[ X(s) = \frac{1}{Ms^2 + Bs + K} F(s) \]

(2\textsuperscript{nd} order system)
Gravity?

At rest, \( \sum F_i = -k\delta + mg = 0 \)

- y coordinate: \( m\ddot{y} = mg - ky \)
- x coordinate: \( m\ddot{x} = mg - k(x + \delta) = -kx \)

Automobile suspension system

\[
\begin{align*}
M_1 x_1''(t) &= -B(x'_1(t) - x'_2(t)) - K_1(x_1(t) - x_2(t)) \\
M_2 x_2''(t) &= f(t) - B(x'_2(t) - x'_1(t)) - K_1(x_2(t) - x_1(t)) - K_2 x_2(t)
\end{align*}
\]
Automobile suspension system

\[
\begin{align*}
M_1 x''_1(t) &= -B(x'_1(t) - x'_2(t)) - K_1(x_1(t) - x_2(t)) \\
M_2 x''_2(t) &= f(t) - B(x'_2(t) - x'_1(t)) - K_1(x_2(t) - x_1(t)) - K_2 x_2(t)
\end{align*}
\]

Laplace transform with zero ICs

\[
\begin{align*}
M_1 s^2 X_1(s) &= -B(sX_1(s) - sX_2(s)) - K_1(X_1(s) - X_2(s)) \\
M_2 s^2 X_2(s) &= F(s) - B(sX_2(s) - sX_1(s)) - K_1(X_2(s) - X_1(s)) - K_2 X_2(s)
\end{align*}
\]

Block diagram

Rotational mechanical elements (constitutive equations)

- Moment of inertia: \( \tau(t) = J \theta''(t) \)
- Rotational spring: \( \tau(t) = K(\theta_1(t) - \theta_2(t)) \)
- Friction: \( \tau(t) = B(\theta'_1(t) - \theta'_2(t)) \)

\[
\begin{align*}
T(s) &= J s^2 \Theta(s) \\
T(s) &= K(\Theta_1(s) - \Theta_2(s)) \\
T(s) &= B s(\Theta_1(s) - \Theta_2(s))
\end{align*}
\]
Torsional pendulum system Ex.2.12

\[ J\theta''(t) + B\theta'(t) + K\theta(t) = \tau(t) \]

- **Equation of Motion**

\[ J\theta''(t) + B\theta'(t) + K\theta(t) = \tau(t) \]

- **By Laplace transform** (with zero ICs),

\[ G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K} \quad (2^{nd} \text{ order system}) \]
Example

- **FBD**

\[
\begin{align*}
\tau_b &= B_m(\dot{\theta}_m - 0) \\
\tau_k &= K(\theta_m - \theta_L)
\end{align*}
\]

- By **Newton’s law**

\[
\begin{align*}
J_m\theta_m''(t) &= \tau_m(t) - B_m\theta_m'(t) - K(\theta_m(t) - \theta_L(t)) \\
J_L\theta_L''(t) &= K(\theta_m(t) - \theta_L(t))
\end{align*}
\]

- By **Laplace transform** (with zero ICs),

\[
\begin{align*}
J_m s^2 \Theta_m(s) &= T_m(s) - B_m s \Theta_m(s) - K(\Theta_m(s) - \Theta_L(s)) \\
J_L s^2 \Theta_L(s) &= K(\Theta_m(s) - \Theta_L(s))
\end{align*}
\]

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Example (cont’d)

- From second equation:
  \[ \Theta_L(s) = \frac{K}{J_L s^2 + K} \Theta_m(s) \]  
  (2nd order system)

- From first equation:
  \[ \Theta_m(s) = \frac{J_L s^2 + K}{s \left( J_m J_L s^2 + B_m J_L s^2 + K (J_m + J_L) s + B_m K \right) G_2(s)} T_m(s) \]  
  (4th order system)

Block diagram

\[ T_m(s) \xrightarrow{G_2} \Theta_m(s) \xrightarrow{G_1} \Theta_L(s) \]

Rigid satellite Ex. 2.13

- Broadcasting
- Weather forecast
- Communication
- GPS, etc.

\[ \tau(t) = J \ddot{\theta}(t) \]

\[ G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2} \]  
Double integrator
Summary & Exercises

- Modeling of mechanical systems
  - Translational
  - Rotational
- Next, block diagrams.
- Exercises
  - Read Sections 2.5, 2.6.
  - Derive equations for the automobile suspension problem.

Exercises (Franklin et al.)

- *Quarter car model*: Obtain a transfer function from $R(s)$ to $Y(s)$.

\[
\frac{Y(s)}{R(s)} = \frac{k_{wb}}{m_1 m_2} \left( s + \frac{k_s}{b} \right) \frac{1}{s^4 + \left( \frac{b}{m_1} + \frac{b}{m_2} \right) s^3 + \left( \frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_{wb}}{m_1 m_2} \right) s^2 + \left( \frac{k_{wb} b}{m_1 m_2} \right) s + \frac{k_{ws}}{m_1 m_2}}
\]

Answer

- Road surface