ME451: Control Systems

Lecture 3
Solution to ODEs via Laplace transform

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems
  • electrical
  • mechanical
  • electromechanical
- Linearization

Analysis
- Time response
  • Transient
  • Steady state
- Frequency response
  • Bode plot
- Stability
  • Routh-Hurwitz
  • Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)
Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function $f(t)$ ($f(t)=0$ for $t<0$),
  \[
  F(s) = \mathcal{L}\{f(t)\} := \int_{0}^{\infty} f(t)e^{-st}dt
  \]
  ($s$: complex variable)

- We denote Laplace transform of $f(t)$ by $F(s)$.

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).

\[
\begin{array}{c}
\text{t-domain} \\
\text{ODE} \xrightarrow{\mathcal{L}} \text{AE} \\
\text{Solution to ODE} \xleftarrow{\mathcal{L}^{-1}} \\
\text{Partial fraction expansion}
\end{array}
\]
Example 1

ODE with initial conditions (ICs)

\[
\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 5u_s(t), \quad y(0) = -1, \quad y'(0) = 2
\]

1. Laplace transform

\[ \mathcal{L}\{y''(t)\} \quad \mathcal{L}\{y'(t)\} \]

Properties of Laplace transform

**Differentiation (review)**

\[ \mathcal{L}\{f'(t)\} = sF(s) - f(0) \]

**t-domain**

\[
\begin{align*}
 f(t) &\rightarrow \frac{d}{dt} & f'(t) &\rightarrow \frac{d}{dt} & f''(t) \\
 \mathcal{L} &\rightarrow & \mathcal{L} &\rightarrow & \mathcal{L}
\end{align*}
\]

**s-domain**

\[
\begin{align*}
 F(s) &\rightarrow S & sF(s) - f(0) &\rightarrow S & s\{sF(s) - f(0)\} - f'(0)
\end{align*}
\]
Example 1 (cont’d)

2. Partial fraction expansion

\[ Y(s) = \frac{-s^2 - s + 5}{s(s + 1)(s + 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2} \]

Multiply both sides by \( s \) & let \( s \) go to zero:

\[ sY(s)_{s \to 0} = A + s\left(\frac{B}{s + 1}\right)_{s \to 0} + s\left(\frac{C}{s + 2}\right)_{s \to 0} \]

Similarly,

Example 1 (cont’d)

3. Inverse Laplace transform

\[ \mathcal{L}^{-1}\left\{ Y(s) = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2} \right\} \]

If we are interested in only the final value of \( y(t) \), apply Final Value Theorem:

\[ \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{-s^2 - s + 5}{(s + 1)(s + 2)} = \frac{5}{2} \]
Example 2

\[ \ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1 \]

- S1
- S2

- S3 \[ y(t) = \mathcal{L}^{-1}(Y(s)) = \]

In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!
Example: Newton’s law

\[ M \frac{d^2 x(t)}{dt^2} = f(t) \]

We want to know the trajectory of \( x(t) \). By Laplace transform,

\[ M \left( s^2 X(s) - sx(0) - x'(0) \right) = F(s) \]

\[ \Rightarrow X(s) = \frac{1}{Ms^2} F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2} \]

(Total response) = (Forced response) + (Initial condition response)

\[ \Rightarrow x(t) = \mathcal{L}^{-1} \left[ \frac{1}{Ms^2} F(s) \right] + x(0)u_s(t) + x'(0)tu_s(t) \]

Ex: Mechanical accelerometer

- Taken from Dorf & Bishop book
Ex: Mechanical accelerometer (cont’d)

- We would like to know how $y(t)$ moves when unit step $f(t)$ is applied with zero ICs.

- By Newton’s law

\[
\begin{align*}
M \frac{d^2}{dt^2}(x(t) + y(t)) &= b \frac{dy(t)}{dt} - ky(t) \\
M_s \frac{d^2 x(t)}{dt^2} &= f(t)
\end{align*}
\]

\[\mathcal{L}\]

\[ Y(s) = \]

Ex: Mechanical accelerometer (cont’d)

- Suppose that $b/M=3$, $k/M=2$ and $M_s=1$.

- Partial fraction expansion

\[ Y(s) = \frac{1}{s^2 + 3s + 2} \cdot \frac{1}{s} = \]

- Inverse Laplace transform

\[ y(t) = \]
Summary & Exercises

- Solution procedure to ODEs
  1. Laplace transform
  2. Partial fraction expansion
  3. Inverse Laplace transform
- Next, modeling of physical systems using Laplace transform
- Exercises
  - Read Appendix B.
  - Derive the solution to the accelerometer problem.
  - Solve Problems:
    • B.3-(a), (b) in page 648
    • B.10 in page 649.