ME451: Control Systems

Lecture 22
Frequency response

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
- Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)
What is frequency response?

- We would like to analyze a system property by applying a *test sinusoidal input* \( u(t) \) and observing a response \( y(t) \).
- Steady state response \( y_{ss}(t) \) (after transient dies out) of a system to sinusoidal inputs is called *frequency response*.

A simple example

- RC circuit
  
  \[
  \begin{align*}
  Y(s) &= \frac{1}{sC}I(s) \\
  U(s) &= \left( R + \frac{1}{sC} \right)I(s) \\
  G(s) &= \frac{1}{RCs + 1}
  \end{align*}
  \]

- Input a sinusoidal voltage \( u(t) \)
- What is the output voltage \( y(t) \)?
An example (cont’d)

- TF (R=C=1)
  \[ G(s) = \frac{1}{s + 1} \]
- \( u(t) = \sin(t) \)

At steady-state, \( u(t) \) and \( y(t) \) has same frequency, but different amplitude and phase!

An example (cont’d)

- Derivation of \( y(t) \)
  \[
  Y(s) = G(s)U(s) = \frac{1}{s + 1} \cdot \frac{1}{s^2 + 1} = \frac{1}{2} \left( \frac{1}{s + 1} + \frac{-s + 1}{s^2 + 1} \right)
  \]
- Inverse Laplace
  \[
  y(t) = \frac{1}{2} \left( e^{-t} - \cos t + \sin t \right)
  \]
  Partial fraction expansion

0 as \( t \) goes to infinity.

\[
 y_{ss}(t) = \frac{1}{2} (-\cos t + \sin t) = \frac{1}{\sqrt{2}} \sin(t - 45^\circ)
\]

(Derivation for general \( G(s) \) is given at the end of lecture slide.)
Response to sinusoidal input

- How is the steady state output of a linear system when the input is sinusoidal?

\[ U(t) = A \sin \omega t \]

\[ y(t) \]

\[ y_{ss}(t) \]

- Steady state output \( y_{ss}(t) = A |G(j\omega)| \sin(\omega t + \angle G(j\omega)) \)
  - Frequency is same as the input frequency \( \omega \)
  - Amplitude is that of input (A) multiplied by \( |G(j\omega)| \)
  - Phase shifts \( \angle G(j\omega) \)

Frequency response function

- For a stable system \( G(s) \), \( G(j\omega) \) (\( \omega \) is positive) is called frequency response function (FRF).
- FRF is a complex number, and thus, has an amplitude and a phase.
- First order example

\[ G(s) = \frac{1}{s + 1} \implies G(j\omega) = \frac{1}{j\omega + 1} \]

\[ |G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2}} \]

\[ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \]
Another example of FRF

- Second order system

\[ G(s) = \frac{2}{s^2 + 3s + 2} \]

\[ G(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{2}{-\omega^2 + j \cdot 3\omega} \]

\[ |G(j\omega)| = \frac{2}{\sqrt{(2 - \omega^2)^2 + 9\omega^2}} \]

\[ \angle G(j\omega) = \angle(2) - \angle(-\omega^2 + j \cdot 3\omega) \]

\[ = -\tan^{-1}\frac{3\omega}{2 - \omega^2} \]

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First order example revisited

- FRF \[ G(j\omega) = \frac{1}{j\omega + 1} \]

| frequency \( \omega \) | amplitude \( |G(j\omega)| \) | phase \( \angle G(j\omega) \) |
|------------------------|-----------------|-----------------|
| 0                      | 1               | 0°              |
| 0.5                    | 0.894           | -26.6°          |
| 1.0                    | 0.707           | -45°            |
| \vdots                 | \vdots          | \vdots          |
| \infty                 | 0               | -90°            |

- Two graphs representing FRF
  - Bode diagram (Bode plot) (Today)
  - Nyquist diagram (Nyquist plot)
Bode diagram (Bode plot) of $G(j\omega)$

- Bode diagram consists of gain plot & phase plot

$$20 \log_{10} |G(j\omega)| \text{ (dB)}$$

$$\angle G(j\omega) \text{ (deg)}$$

Bode plot of a 1st order system

- TF

$$G(s) = \frac{1}{Ts + 1}$$

$$G(j\omega) = \frac{1}{j\omega T + 1}$$

$$\approx \begin{cases} 
1 & \text{if } 1 \gg \omega T \\
\frac{1}{j\omega T} & \text{if } 1 \ll \omega T 
\end{cases}$$
Exercises of sketching Bode plot

- First order system

\[ G(s) = \frac{1}{s + 1} \quad G(s) = \frac{1}{0.1s + 1} \quad G(s) = \frac{1}{10s + 1} \]

Remarks on Bode diagram

- Bode diagram shows amplification and phase shift of a system output for sinusoidal inputs with various frequencies.
- It is very useful and important in analysis and design of control systems.
- The shape of Bode plot contains information of stability, time responses, and much more!
- It can also be used for system identification. (Given FRF experimental data, obtain a transfer function that matches the data.)
System identification

- Sweep frequencies of sinusoidal signals and obtain FRF data (i.e., gain and phase).
- Select $G(s)$ so that $G(j\omega)$ fits the FRF data.

**Agilent Technologies: FFT Dynamic Signal Analyzer**

Generate sin signals
Sweep frequencies

Collect FRF data
Select $G(s)$

Unknown system

Summary and exercises

- Frequency response is a steady state response of systems to a sinusoidal input.
- For a linear system, sinusoidal input generates sinusoidal output with **same frequency** but **different amplitude and phase**.
- Bode plot is a graphical representation of frequency response function. ("bode.m")
- Next, Bode diagram of simple transfer functions
- Exercise: Read Section 8.
Derivation of frequency response

\[ Y(s) = G(s)U(s) = G(s) \frac{A\omega}{s^2 + \omega^2} = \frac{k_1}{s + j\omega} + \frac{k_2}{s - j\omega} + C_g(s) \]

\[ k_1 = \lim_{s \to -j\omega} (s + j\omega)G(s)\frac{A\omega}{s^2 + \omega^2} = G(-j\omega)\frac{A\omega}{-2j\omega} = -\frac{AG(-j\omega)}{2j} \]

\[ k_2 = \lim_{s \to j\omega} (s - j\omega)G(s)\frac{A\omega}{s^2 + \omega^2} = G(j\omega)\frac{A\omega}{2j\omega} = \frac{AG(j\omega)}{2j} \]

\[ y(t) = k_1 e^{-j\omega t} + k_2 e^{j\omega t} + \mathcal{L}^{-1}\{C_g(s)\} \]

\[ y_{ss}(t) = A|G(j\omega)|\frac{e^{j(\omega t + \angle G(j\omega))} - e^{-j(\omega t + \angle G(j\omega))}}{2j} \]

\[ \sin(\omega t + \angle G(j\omega)) \]