ME451: Control Systems

Lecture 13
Steady-state error

Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems:
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis
- Time response:
  - Transient
  - Steady state
- Frequency response:
  - Bode plot
- Stability:
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations &) laboratories
Performance measures (review)

- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error

Next, we will connect these measures with s-domain.

Steady-state error: unity feedback

We assume that the CL system is stable!

Unity feedback!

- Suppose that we want output $y(t)$ to track $r(t)$.
- Error $e(t) = r(t) - y(t)$
- Steady-state error
  \[ e_{ss} := \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + G(s)} R(s) \]

Final value theorem
(Suppose CL system is stable!!!)
Error constants

- Step-error (position-error) constant
  \[ K_p := \lim_{s \to 0} G(s) \]

- Ramp-error (velocity-error) constant
  \[ K_v := \lim_{s \to 0} sG(s) \]

- Parabolic-error (acceleration-error) constant
  \[ K_a := \lim_{s \to 0} s^2G(s) \]

- \( K_p, K_v, K_a \): ability to reduce steady-state error

Steady-state error for step \( r(t) \)

\[ r(t) = Ru_s(t) \Rightarrow e_{ss} = \frac{R}{1 + K_p} \]
Steady-state error for ramp \( r(t) \)

\[
r(t) = R u_s(t) \Rightarrow e_{ss} = \frac{R}{K_v}
\]

Steady-state error for parabolic \( r(t) \)

\[
r(t) = \frac{Rt^2}{2} u_s(t) \Rightarrow e_{ss} = \frac{R}{K_a}
\]
System type

- **System type of G** is defined as the order (number) of poles of G(s) at s=0.
- **Examples**

  \[ G(s) = \frac{K(1 + 0.5s)}{s(1 + s)(1 + 2s)(1 + s + s^2)} \rightarrow \text{type 1} \]

  \[ G(s) = \frac{K(1 + s)}{s^2}e^{-Ts} \rightarrow \text{type 2} \]

  \[ G(s) = \frac{K(1 + 2s)}{s^3} \rightarrow \text{type 3} \]

Zero steady-state error

- If error constant is infinite, we can achieve zero steady-state error. (Accurate tracking)
  - For step r(t)
    \[ K_p = \lim_{{s \to 0}} G(s) = \infty \Leftrightarrow G(s) \text{ is of at least type 1} \]
  - For ramp r(t)
    \[ K_v = \lim_{{s \to 0}} sG(s) = \infty \Leftrightarrow G(s) \text{ is of at least type 2} \]
  - For parabolic r(t)
    \[ K_a = \lim_{{s \to 0}} s^2G(s) = \infty \Leftrightarrow G(s) \text{ is of at least type 3} \]
Example 1

- $G(s)$ of type 2
  \[ G(s) = \frac{K}{s^2(s + 12)} \]

- Characteristic equation
  \[ 1 + G(s) = 0 \Leftrightarrow s^2(s + 12) + K = 0 \Leftrightarrow s^3 + 12s^2 + K = 0 \]

- CL system is NOT stable for any $K$.
- $e(t)$ goes to infinity. (Don’t use today’s results if CL system is not stable!!)

Example 2

- $G(s)$ of type 1
  \[ G(s) = \frac{K(s + 3.15)}{s(s + 1.5)(s + 0.5)} \]

- By Routh-Hurwitz criterion, CL is stable iff
  \[ 0 < K < 1.304 \]

- Step $r(t)$
  \[ e_{ss} = \frac{R}{1 + K_p} = 0 \]

- Ramp $r(t)$
  \[ e_{ss} = \frac{R}{K_v}, \quad K_v := \lim_{s \to 0} sG(s) = \frac{3.15K}{0.75} = 4.2K \]

- Parabolic $r(t)$
  \[ e_{ss} = \frac{R}{K_a} = \infty, \quad K_a := \lim_{s \to 0} s^2 G(s) = 0 \]
Example 3

- G(s) of type 2
  \[ G(s) = \frac{5(s + 1)}{s^2(s + 12)(s + 5)} \]
  - By Routh-Hurwitz criterion, we can show that CL system is stable.
- Step r(t)
  \[ e_{ss} = \frac{R}{1 + K_p} = 0 \]
- Ramp r(t)
  \[ e_{ss} = \frac{R}{K_v} = 0 \]
- Parabolic r(t)
  \[ e_{ss} = \frac{R}{K_a} = 12R \quad K_a := \lim_{s \to 0} s^2G(s) = \frac{1}{12} \]

A control example

- Closed-loop stable?
- Compute error constants
  \[ K_p = \quad K_v = \quad K_a = \]
- Compute steady state errors
  \[ e_{ss} = \quad e_{ss} = \quad e_{ss} = \]
Summary and Exercises

- **Steady-state error**
  - For *unity feedback* (STABLE!) systems, the system type of the forward-path system determines if the steady-state error is zero.
  - The key tool is the final value theorem!

- **Next, time response of 1st-order systems**

- **Exercises**
  - Read Section 5.5.
  - Solve Problems 5.9 and 5.14.