ME451: Control Systems

Lecture 10
Routh-Hurwitz stability criterion

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)
Stability summary (review)

Let $s_i$ be poles of rational $G$. Then, $G$ is …

- **(BIBO, asymptotically) stable** if $\text{Re}(s_i) < 0$ for all $i$.
- **marginally stable** if
  - $\text{Re}(s_i) \leq 0$ for all $i$, and
  - simple root for $\text{Re}(s_i) = 0$
- **unstable** if
  - it is neither stable nor marginally stable.

### Routh-Hurwitz criterion

- This is for LTI systems with a *polynomial* denominator (without sin, cos, exponential etc.).
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does **NOT** explicitly compute the roots.
- No proof is provided in any control textbook.
Polynomial and an assumption

- Consider a polynomial
  \[ Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \]

- Assume \( a_0 \neq 0 \)
  - If this assumption does not hold, \( Q \) can be factored as
    \[ Q(s) = s^m \left( \hat{a}_{n-m} s^{n-m} + \cdots + \hat{a}_1 s + \hat{a}_0 \right) \]
    where \( \hat{a}_0 \neq 0 \)
  - The following method applies to the polynomial \( \hat{Q}(s) \)

Routh array

From the given polynomial

<table>
<thead>
<tr>
<th>( s^n )</th>
<th>( a_n )</th>
<th>( a_{n-2} )</th>
<th>( a_{n-4} )</th>
<th>( a_{n-6} )</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^{n-1} )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-5} )</td>
<td>( a_{n-7} )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^{n-2} )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>( b_4 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^{n-3} )</td>
<td>( c_1 )</td>
<td>( c_2 )</td>
<td>( c_3 )</td>
<td>( c_4 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( s^2 )</td>
<td>( k_1 )</td>
<td>( k_2 )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( l_1 )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( m_1 )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>
Routh array
(How to compute the third row)

\[ \begin{array}{cccccc}
\text{s}^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
\text{s}^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
\text{s}^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
\text{s}^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\text{s}^2 & k_1 & k_2 & & & \\
\text{s}^1 & l_1 & & & & \\
\text{s}^0 & m_1 & & & & \\
\end{array} \]

- \[ b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \]
- \[ b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \]
- \[ \vdots \]

Routh array
(How to compute the fourth row)

\[ \begin{array}{cccccc}
\text{s}^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
\text{s}^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
\text{s}^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
\text{s}^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\text{s}^2 & k_1 & k_2 & & & \\
\text{s}^1 & l_1 & & & & \\
\text{s}^0 & m_1 & & & & \\
\end{array} \]

- \[ c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} \]
- \[ c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} \]
- \[ \vdots \]
Routh-Hurwitz criterion

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.

Example 1

\[ Q(s) = s^3 + s^2 + 2s + 8 \quad (= (s + 2)(s^2 - s + 4)) \]

Routh array

\[
\begin{array}{ccc}
 s^3 & 1 & 2 \\
 s^2 & 1 & 8 \\
 s^1 & -6 \\
 s^0 & 8 \\
\end{array}
\]

Two sign changes in the first column

Two roots in RHP

\[ \frac{1}{2} \pm \frac{j\sqrt{15}}{2} \]
Example 2

\[ Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 \]

Routh array

\[
\begin{array}{c|ccc}
 s^5 & 1 & 2 & 11 \\
 s^4 & 2 & 4 & 10 \\
 s^3 & \varepsilon & 6 & \varepsilon \\
 s^2 & \frac{4\varepsilon - 12}{\varepsilon} & 10 & \frac{\varepsilon}{\varepsilon} < 0 \\
 s^1 & 6 & \varepsilon & \varepsilon \\
 s^0 & 10 & \varepsilon & \varepsilon \\
\end{array}
\]

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number. In this case, \( Q \) has some roots in RHP.

Two sign changes in the first column \( \rightarrow \) Two roots in RHP

\( \varepsilon \rightarrow \frac{4\varepsilon - 12}{\varepsilon} \rightarrow 6 \)

Example 3

\[ Q(s) = s^4 + s^3 + 3s^2 + 2s + 2 \]

Routh array

\[
\begin{array}{c|ccc}
 s^4 & 1 & 3 & 2 \\
 s^3 & 1 & 2 & \varepsilon \\
 s^2 & 1 & 2 & \varepsilon \\
 s^1 & \varepsilon & 2 & \varepsilon \\
 s^0 & 2 & \varepsilon & \varepsilon \\
\end{array}
\]

If zero row appears in Routh array, \( Q \) has roots either on the imaginary axis or in RHP.

No sign changes in the first column \( \rightarrow \) No roots in RHP

But some roots are on imaginary axis.

Take derivative of an auxiliary polynomial (which is a factor of \( Q(s) \))

\[ s^2 + 2 \]
Example 4

\[ Q(s) = s^3 + 3Ks^2 + (K + 2)s + 4 \]

Find the range of \( K \) s.t. \( Q(s) \) has all roots in the left half plane. (Here, \( K \) is a design parameter.)

Routh array

| \( s^3 \) | 1 \( \quad \) \( K + 2 \) \\
| \( s^2 \) | 3\( K \) \( \quad \) 4 \\
| \( s^1 \) | \( \frac{3K(K+2)-4}{3K} \) \\
| \( s^0 \) | 4

No sign changes in the first column

\[ \begin{align*} 3K &> 0 \\
3K(K + 2) - 4 &> 0 \\
K &> -1 + \frac{\sqrt{21}}{3} \end{align*} \]

Simple & important criteria for stability

- **1\(^{st}\) order polynomial** \( Q(s) = a_1s + a_0 \)
  
  All roots are in LHP \( \Leftrightarrow \) \( a_1 \) and \( a_0 \) have the same sign

- **2\(^{nd}\) order polynomial** \( Q(s) = a_2s^2 + a_1s + a_0 \)
  
  All roots are in LHP \( \Leftrightarrow \) \( a_2 \), \( a_1 \) and \( a_0 \) have the same sign

- **Higher order polynomial** \( Q(s) = a_ns^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \)
  
  All roots are in LHP \( \Rightarrow \) All \( a_k \) have the same sign
### Examples

<table>
<thead>
<tr>
<th>( Q(s) )</th>
<th>All roots in open LHP?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3s + 5 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( -2s^2 - 5s - 100 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( 523s^2 - 57s + 189 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>((s^2 + s - 1)(s^2 + s + 1))</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( s^3 + 5s^2 + 10s - 3 )</td>
<td>Yes / No</td>
</tr>
</tbody>
</table>

### Summary and Exercises

- **Routh-Hurwitz stability criterion**
  - **Routh array**
  - Routh-Hurwitz criterion is applicable to only polynomials (so, it is not possible to deal with exponential, sin, cos etc.).

- **Next,**
  - Routh-Hurwitz criterion in control examples

- **Exercises**
  - Read Section 6.
  - Do Examples and Problems 6-2.