Time Response*, ME451

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* This presentation is created by Jongeun Choi and Gabrial Gomes

Zeros and poles of a transfer function

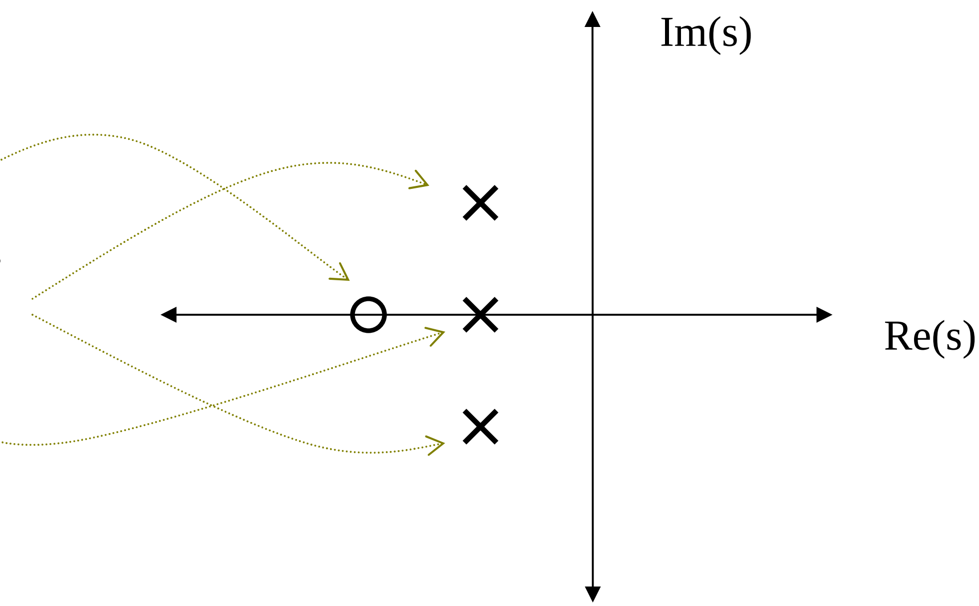
- Let G(s)=N(s)/D(s), then
  - Zeros of G(s) are the roots of N(s)=0
  - Poles of G(s) are the roots of D(s)=0

$$G(s) = \frac{(s+2)}{(s+1+j)(s+1-j)(s+1)}$$

$$z_1 = -2$$

$$p_{1,2} = -1 \pm j$$

$$p_3 = -1$$



Theorems

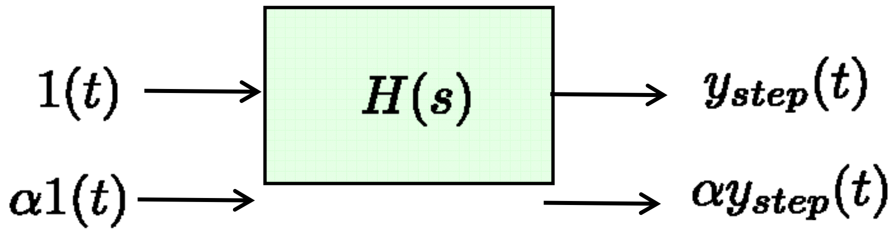
- Initial Value Theorem

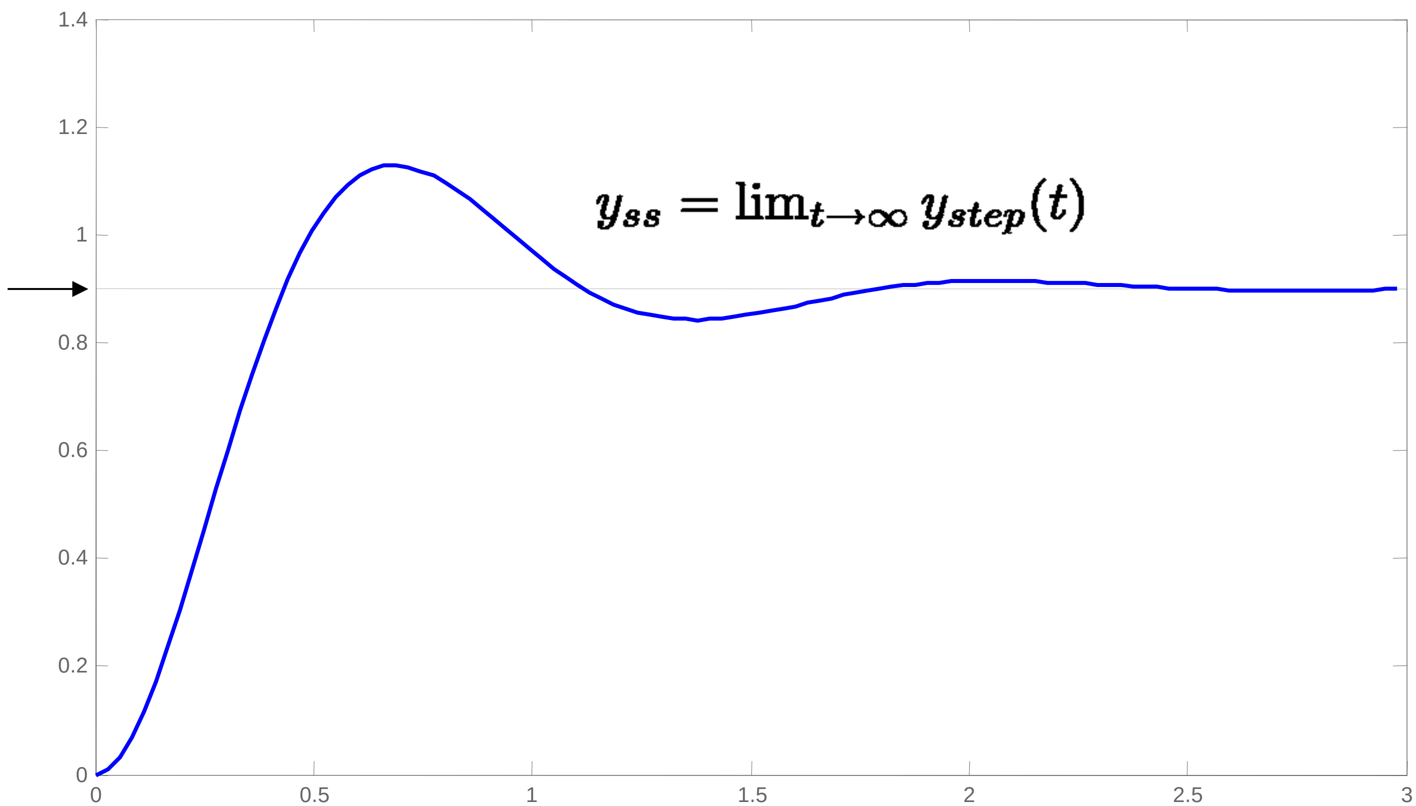
$$\lim_{t \to 0} x(t) = \lim_{s \to \infty} s\mathcal{L}(x(t))$$

- Final Value Theorem
  - If all poles of sX(s) are in the left half plane (LHP), then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\mathcal{L}(x(t))$$

DC gain or static gain of a stable system





DC Gain of a stable transfer function $G(s)$

- DC gain (static gain) : the ratio of the steady state output of a system to its constant input, i.e., steady state of the unit step response
- Use final value theorem to compute the steady state of the unit step response

$$\mathcal{L}(y_{step}(t)) = G(s)\frac{1}{s}$$

$$\begin{aligned}
\text{DC gain} &= \lim_{t\to\infty} y_{step}(t) \\
&= \lim_{s\to 0} \underbrace{s\left[G(s)\frac{1}{s}\right]}_{\text{all poles in LHP} \leftrightarrow G(s) \text{ is stable}} \\
&= \lim_{s\to 0} G(s)
\end{aligned}$$

$$\boxed{\text{DC gain} = \lim_{s\to 0} G(s)}$$

Pure integrator

$$u(t) \longrightarrow \boxed{y(0) + \int_0^t (\cdot)dt} \longrightarrow y(t)$$

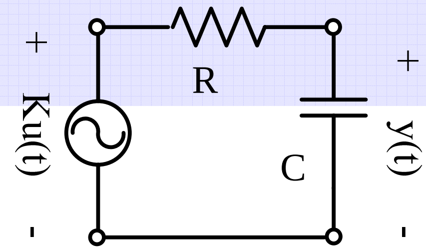
- ODE : $\dot{y}(t) = u(t) \rightarrow G(s) = \dfrac{1}{s}$
- Impulse response : $g(t) = \mathcal{L}^{-1}[G(s)1] = \mathcal{L}^{-1}[1/s] = 1(t)$
- Step response : $y_{step}(t) = \mathcal{L}^{-1}[G(s)1/s] = \mathcal{L}^{-1}[1/s^2] = t1(t)$
- If the initial condition is not zero, then :

$$\dot{y}(t) = u(t) \xrightarrow{\mathcal{L}} sY(s) - y(0) = U(s)$$

$$y(t) = \mathcal{L}^{-1}\left[y(0)\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{U(s)}{s}\right] = y(0) + \mathcal{L}^{-1}(G(s)U(s))$$

Physical meaning of the impulse response

First order system



$$\tau = RC$$

- ODE : $\tau \dot{y}(t) + y(t) = Ku(t) \rightarrow G(s) = \dfrac{K}{\tau s + 1}$

- Impulse response :

$$g(t) = \mathcal{L}^{-1}[G(s)1] = \mathcal{L}^{-1}\left[\frac{K/\tau}{s + 1/\tau}\right] = \frac{K}{\tau} e^{-t/\tau} 1(t)$$

- Step response :

$$y_{step}(t) = \mathcal{L}^{-1}\left[\frac{K}{(\tau s + 1)s}\right] = \mathcal{L}^{-1}\left[K\left(\frac{1}{s} - \frac{\tau}{\tau s + 1}\right)\right]$$
$$= K[1 - e^{-t/\tau}]1(t)$$

- DC gain: (Use the final value theorem)

$$y_{ss} = \lim_{t \to \infty} y_{step}(t) = \lim_{s \to 0} \frac{K}{(\tau s + 1)s} s = K$$

First order system

- If the initial condition was not zero, then

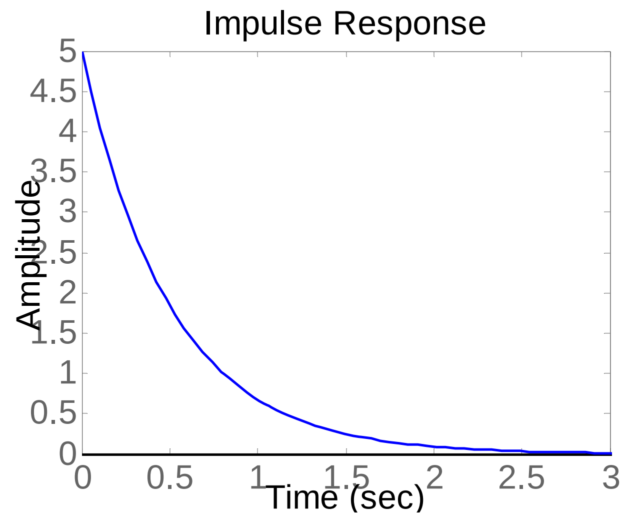
$$\tau \dot{y}(t) + y(t) = Ku(t) \xrightarrow{\mathcal{L}} \tau s Y(s) - \underbrace{y(0)}_{\neq 0} + Y(s) = KU(s)$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[(y(0)/K + U(s))\frac{K}{(\tau s + 1)}\right] \\ &= \mathcal{L}^{-1}\left[(y(0)/K + U(s))G(s)\right] \\ &= \underbrace{\frac{y(0)}{K}\mathcal{L}^{-1}(G(s))}_{\text{from the initial condition}} + \underbrace{\mathcal{L}^{-1}(G(s)U(s))}_{\text{from the input}} \end{aligned}$$

Physical meaning of the impulse response

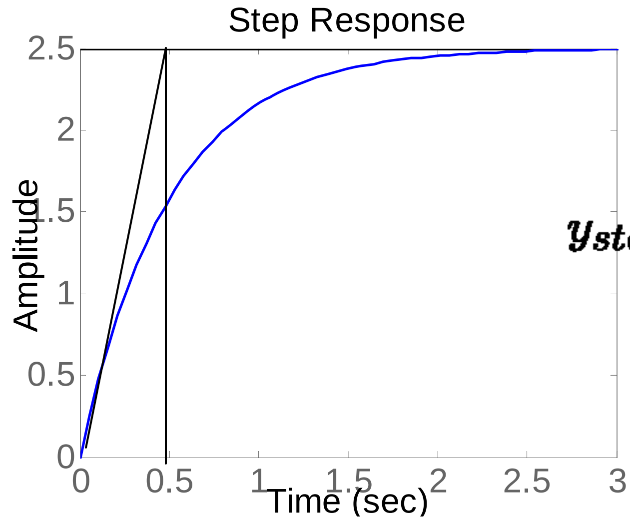
Matlab Simulation $G(s) = \frac{5}{s+2}$

- G=tf([0 5],[1 2]);
- impulse(G)



$$g(t) = \frac{K}{\tau} e^{-t/\tau} 1(t)$$

- step(G)



$$y_{step}(t) = K[1 - e^{-t/\tau}]1(t)$$

- Time constant

$$\tau = 0.5$$

First order system response

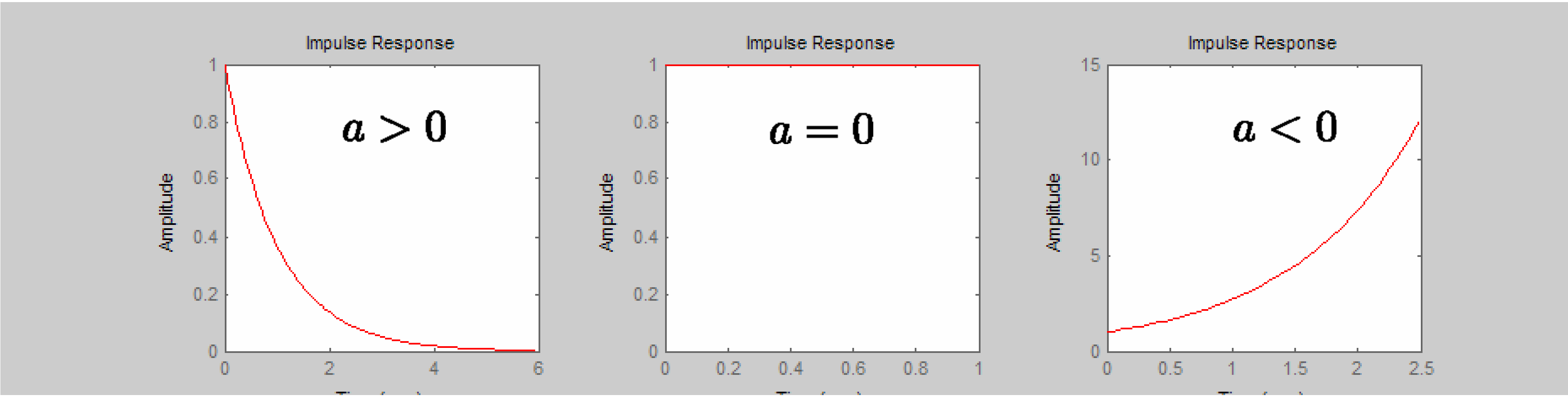
- System transfer function : $$H(s) = \frac{b}{s+a}$$

First order system response

- System transfer function : $H(s) = \frac{b}{s+a}$
- Impulse response : $h(t) \;=\; \mathcal{L}^{-1}[H(s)] \;=\; b\,e^{-at}\,1(t)$

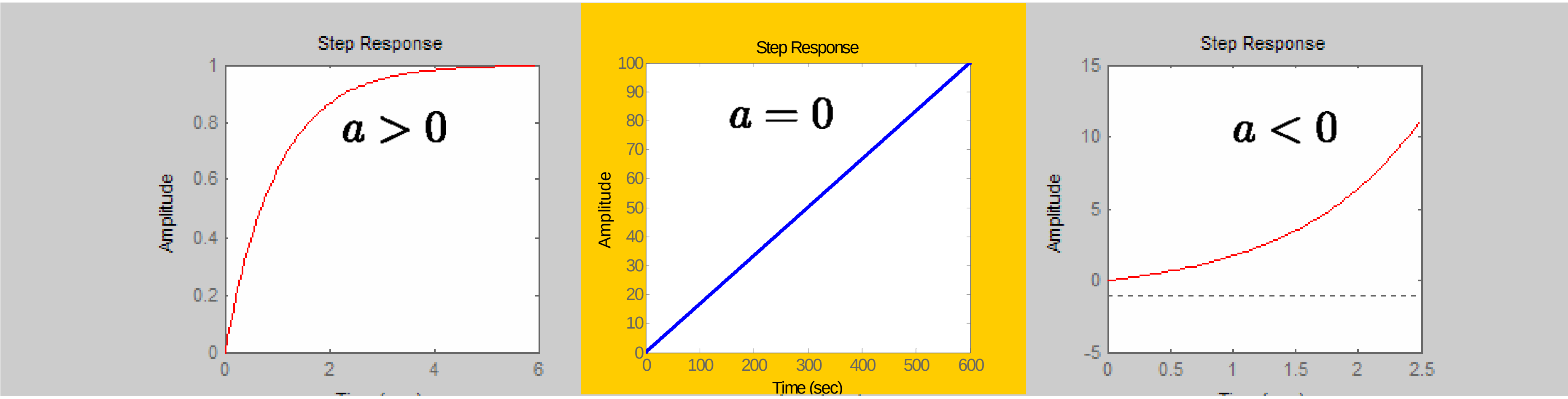
First order system response

- System transfer function : $H(s) = \dfrac{b}{s+a}$
- Impulse response : $h(t) = \mathcal{L}^{-1}[H(s)] = b\,e^{-at}\,1(t)$

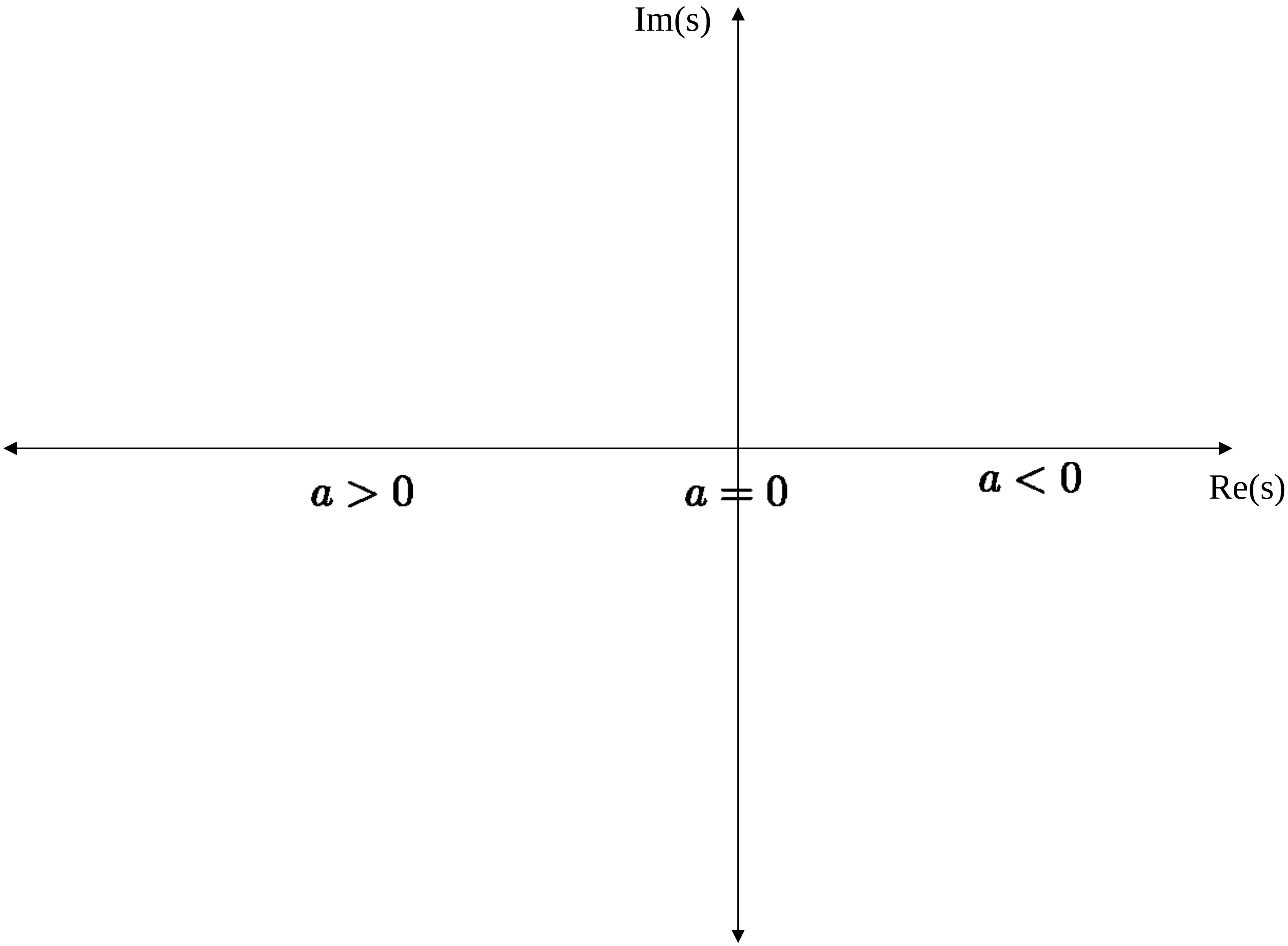


First order system response

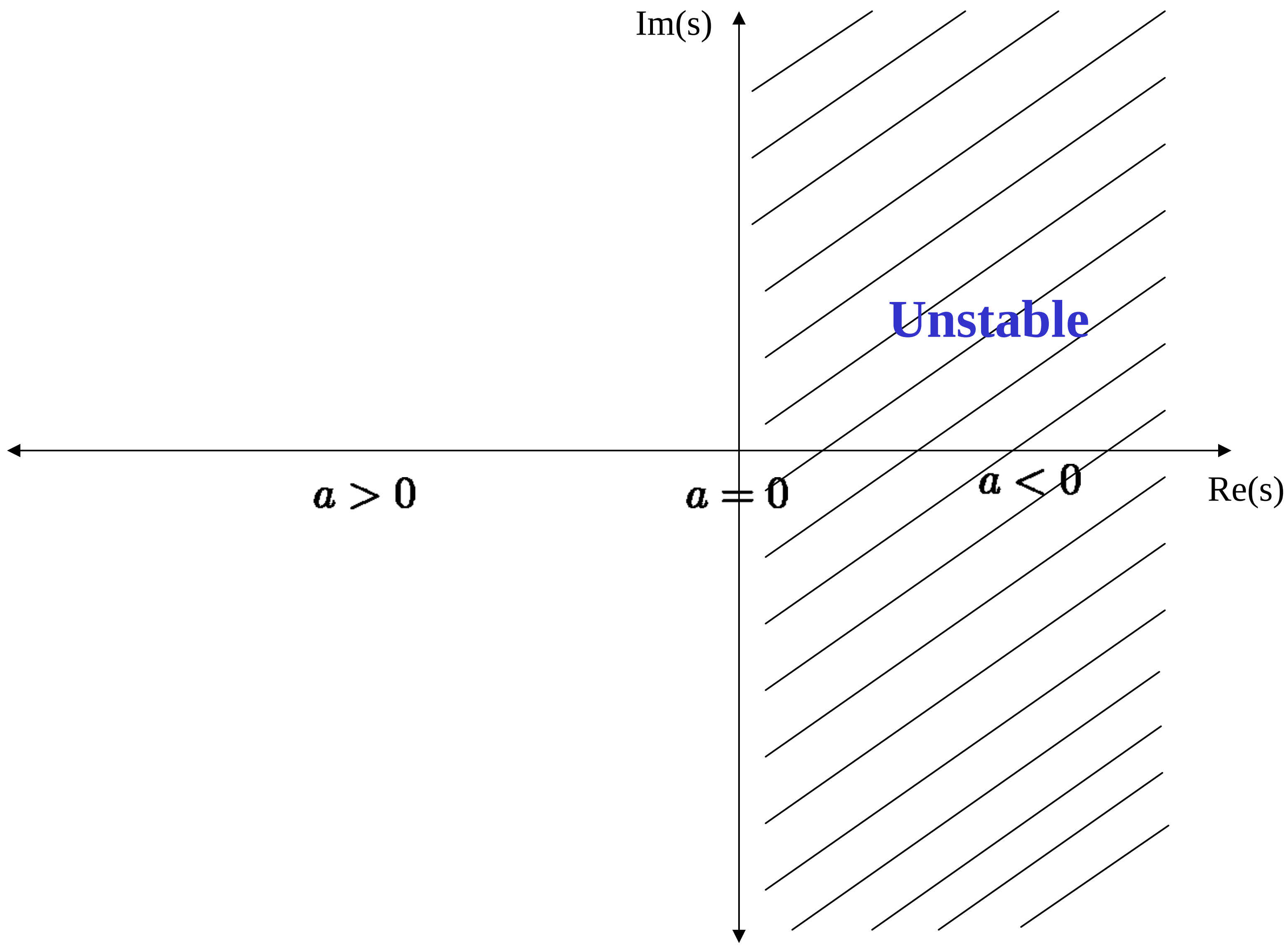
- System transfer function : $H(s) = \dfrac{b}{s+a}$
- Impulse response : $h(t) = \mathcal{L}^{-1}[H(s)] = b\, e^{-at}\, 1(t)$
- Step response : $y_{step}(t) = \mathcal{L}^{-1}[H(s)/s] = \dfrac{b}{a}(1 - e^{-at}) 1(t)$



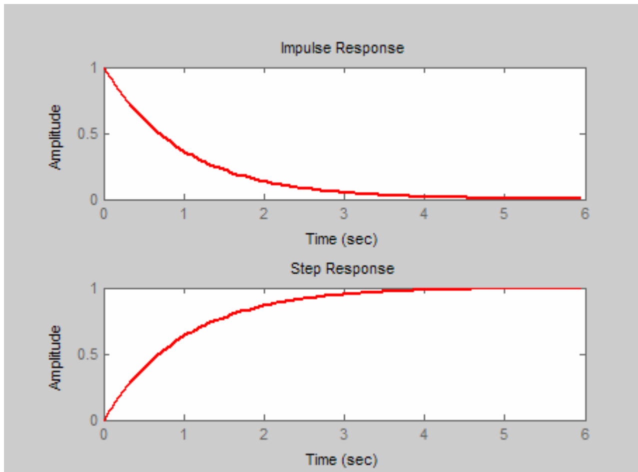
First order system response

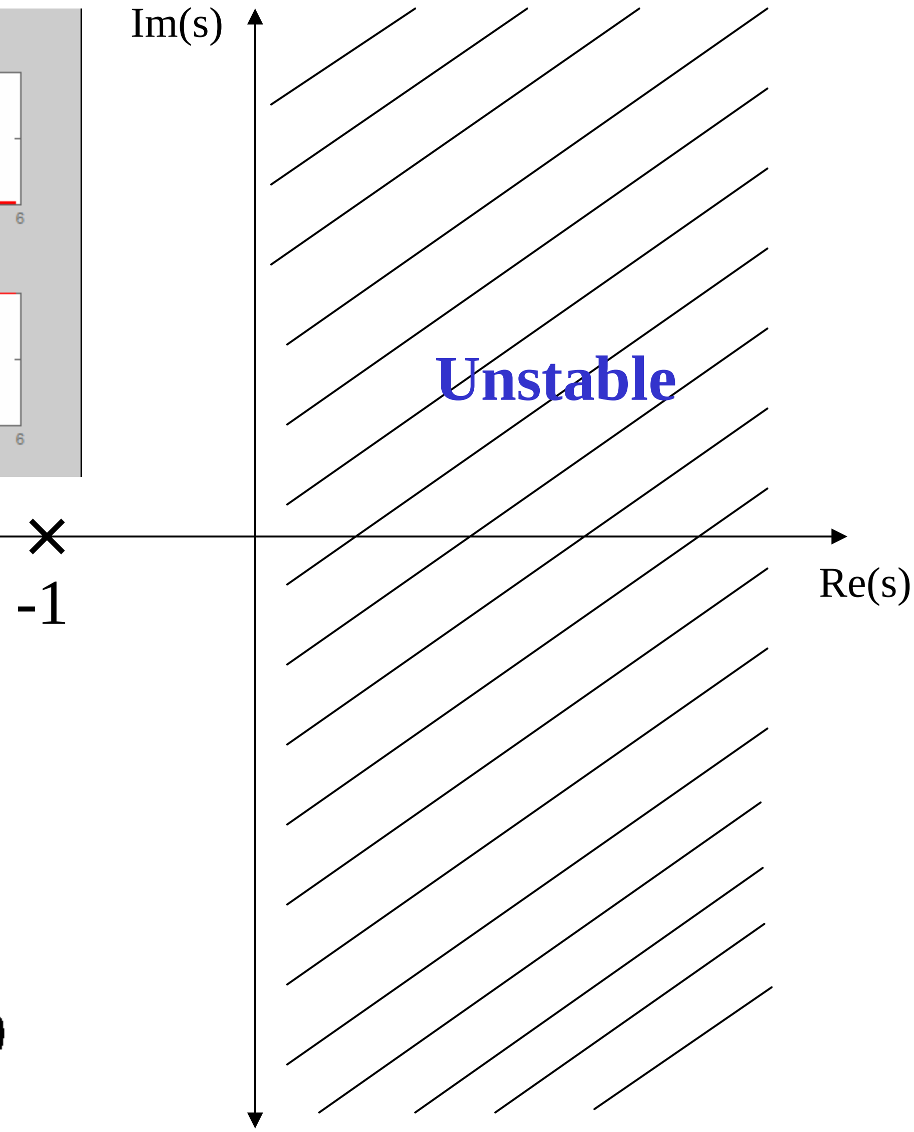


First order system response



First order system response

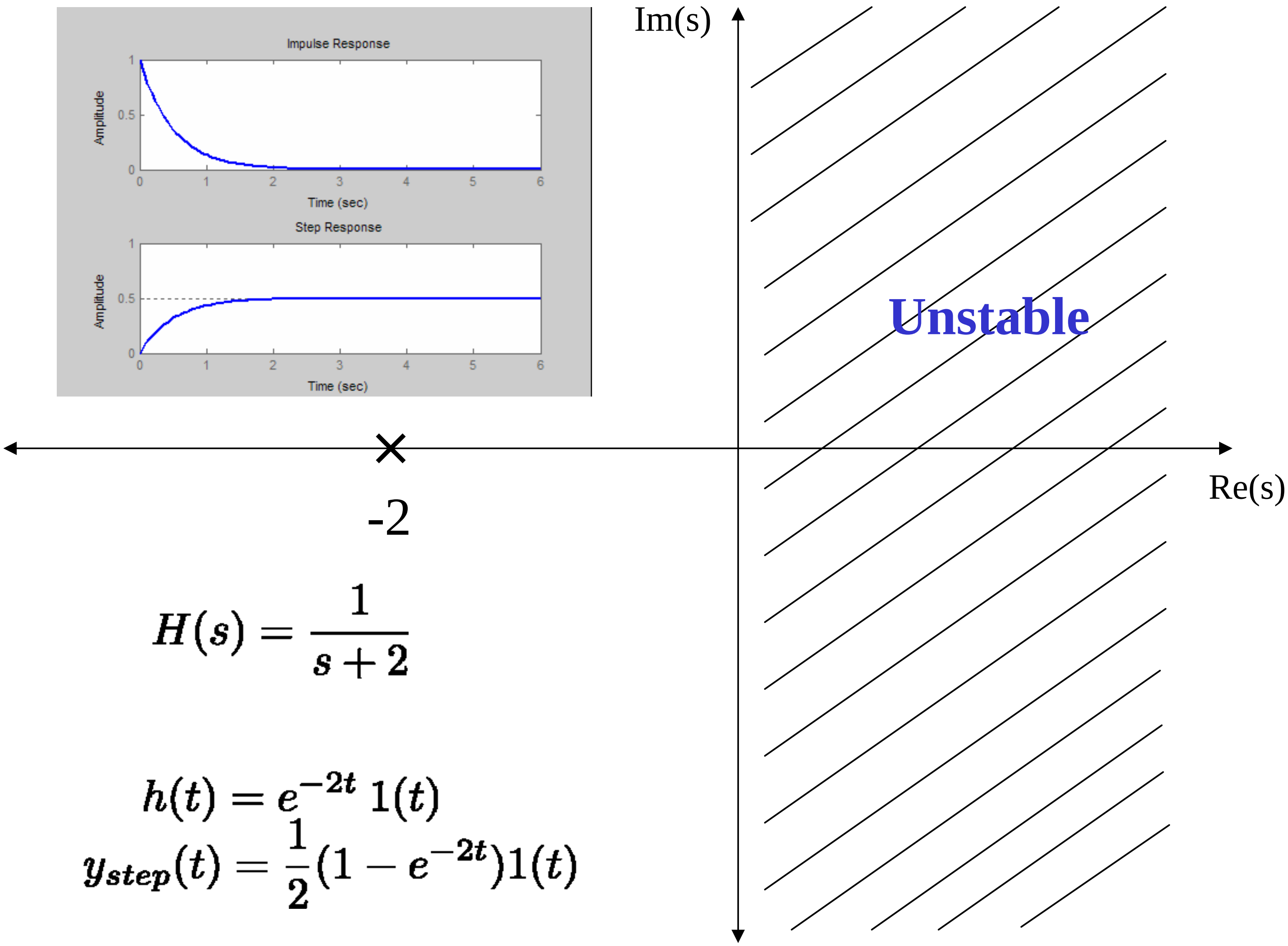




$$H(s) = \frac{1}{s+1}$$

$$h(t) = e^{-t}\,1(t)$$
$$y_{step}(t) = (1 - e^{-t})1(t)$$

First order system response

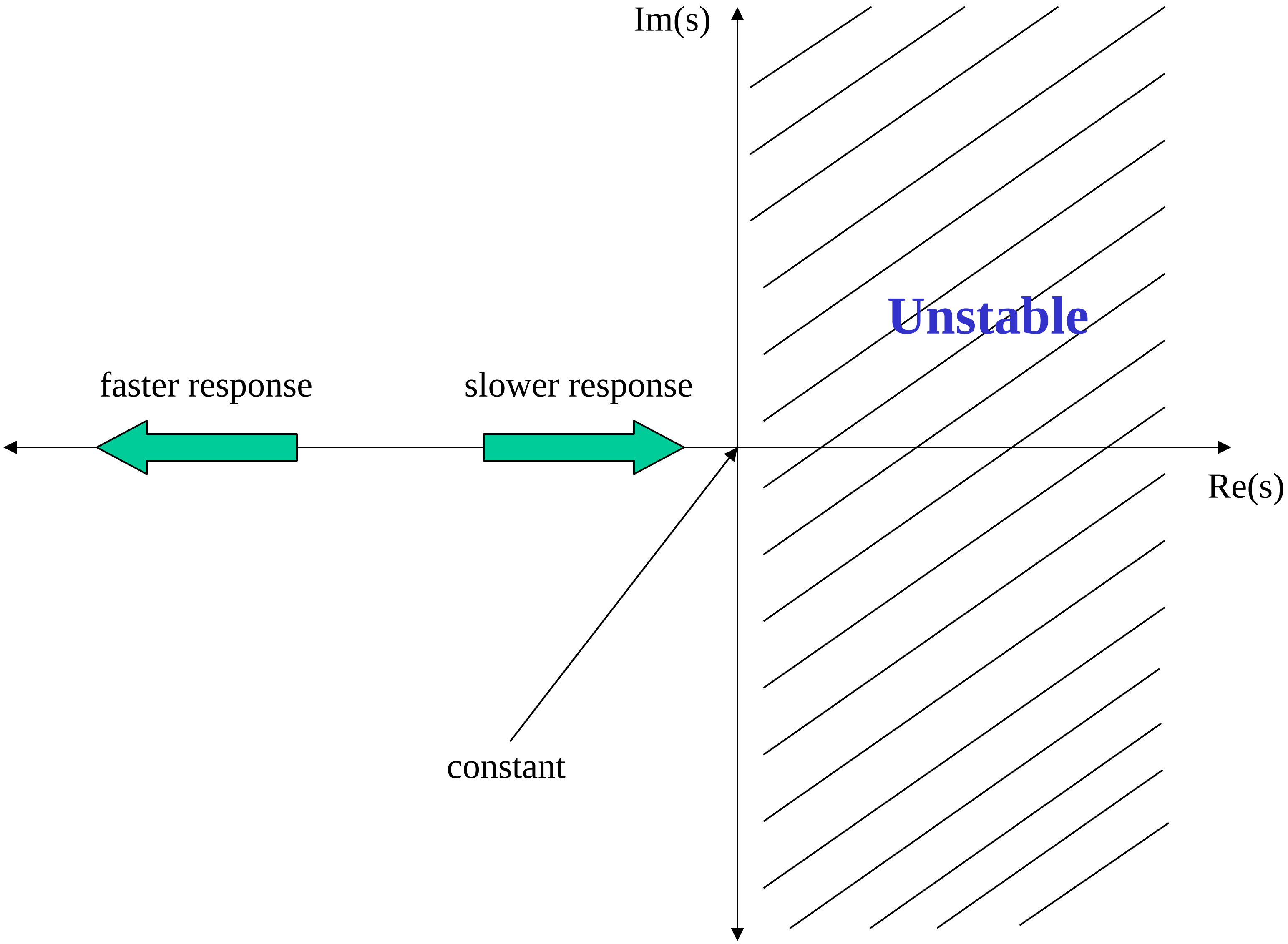


$$H(s) = \frac{1}{s+2}$$

$$h(t) = e^{-2t}\,1(t)$$

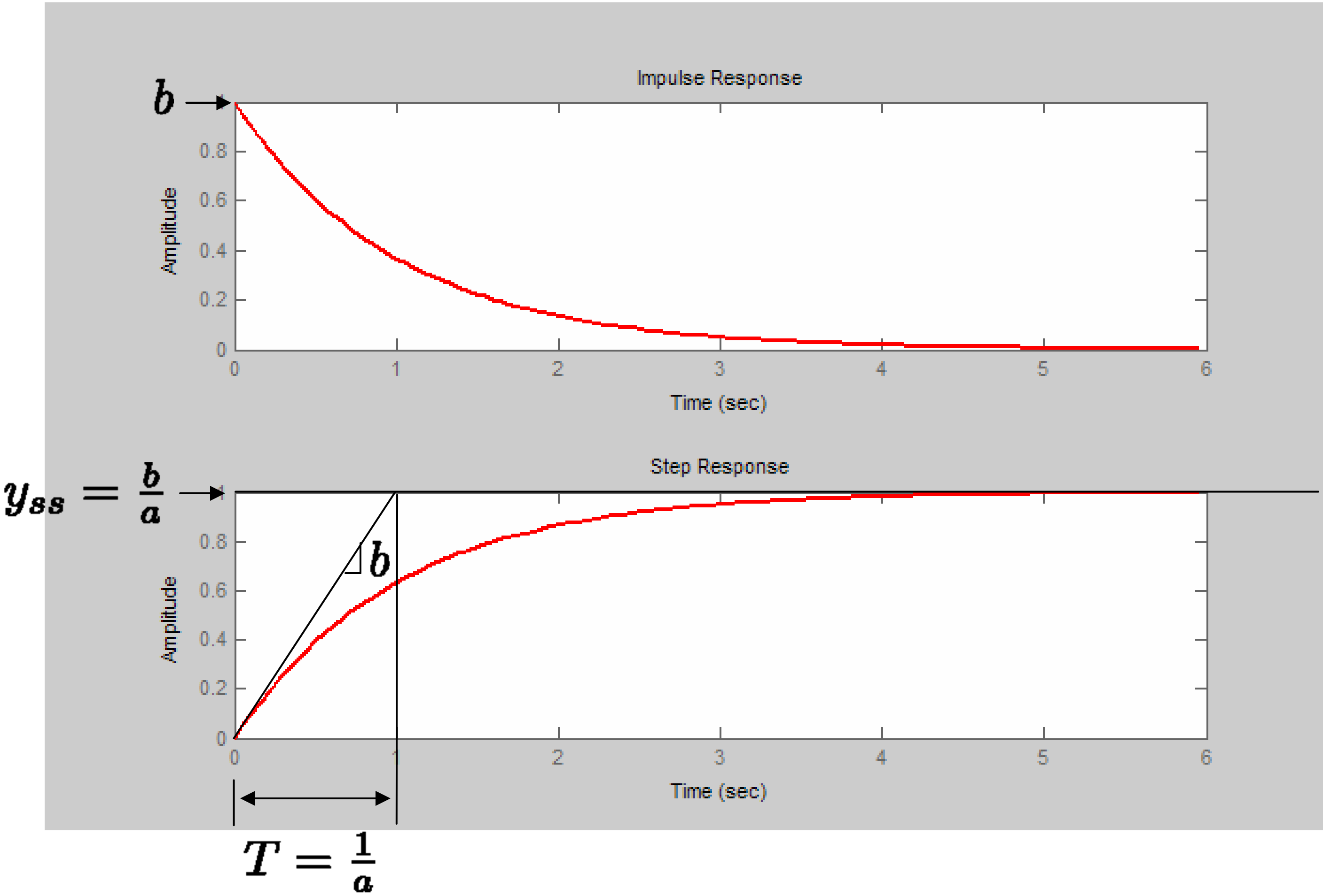
$$y_{step}(t) = \frac{1}{2}(1 - e^{-2t})1(t)$$

First order system response



First order system – Time specifications.

$$H(s) = \frac{b}{s+a}$$



First order system – Time specifications.

$$H(s) = \frac{b}{s+a}$$

Time specs:

- Steady state value : $y_{ss} = \lim_{t\to\infty} y_{step}(t) = \frac{b}{a}$
- Time constant : $T = \frac{1}{a}$ $\quad y_{step}(T) = 0.63\, y_{ss}$
- Rise time : $T_r = \frac{2.2}{a}$ $\quad$ Time to go from $0.1\, y_{ss}$ to $0.9\, y_{ss}$
- Settling time : $T_s = \frac{4}{a}$ $\quad y_{step}(T_s) = 0.98\, y_{ss}$

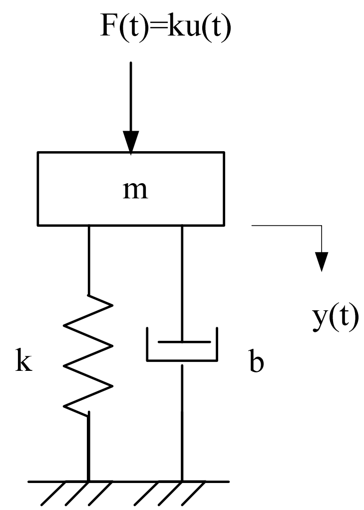
First order system – Simple behavior.

$$H(s) = \frac{b}{s+a}$$

- No overshoot
- No oscillations

Second order system (mass-spring-damper system)

- ODE : $$\ddot{y}(t) + \frac{b}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{k}{m}u(t)$$



ζ : dampling ratio, ω_n : natural frequency

$$2\zeta\omega_n = b/m,\ \omega_n^2 = k/m$$

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t)$$

- Transfer function :

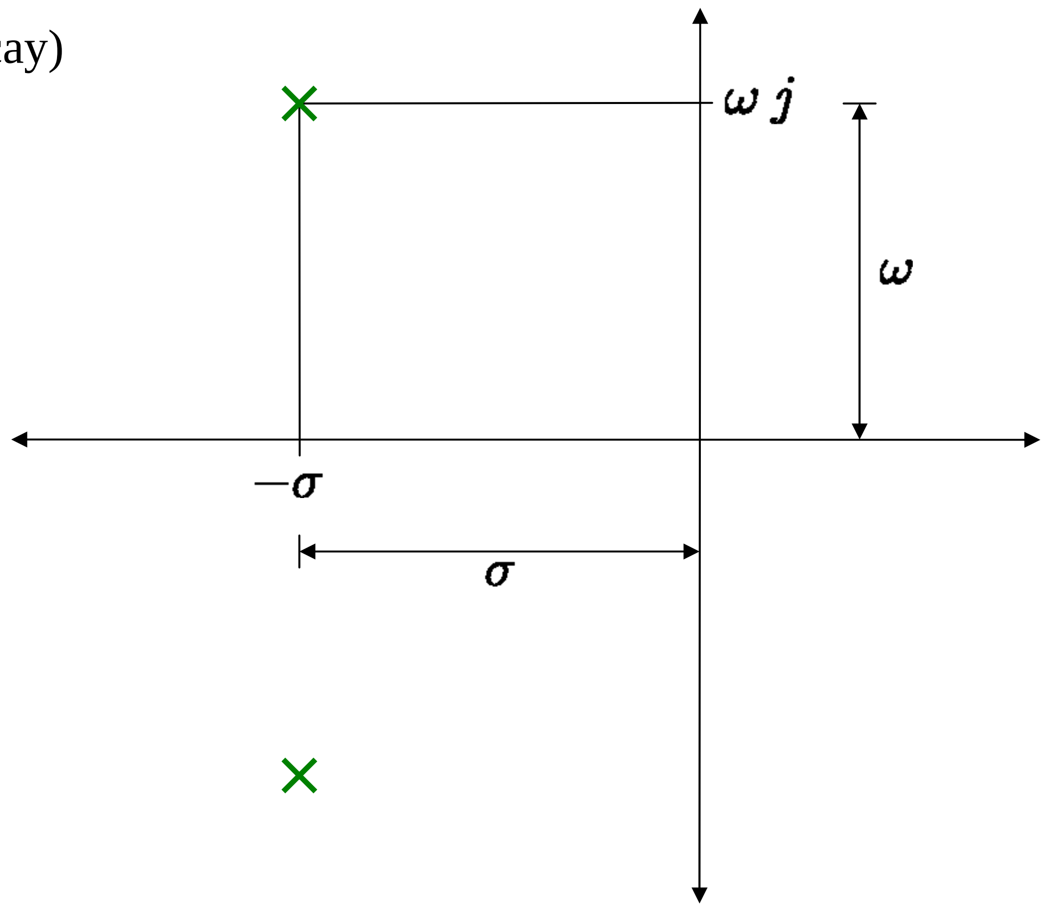
$$\frac{Y(s)}{U(s)} = H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n^2 s + \omega_n^2}$$

Polar vs. Cartesian representations.

Cartesian representation :

ω … Imaginary part (frequency)

σ … Real part (rate of decay)



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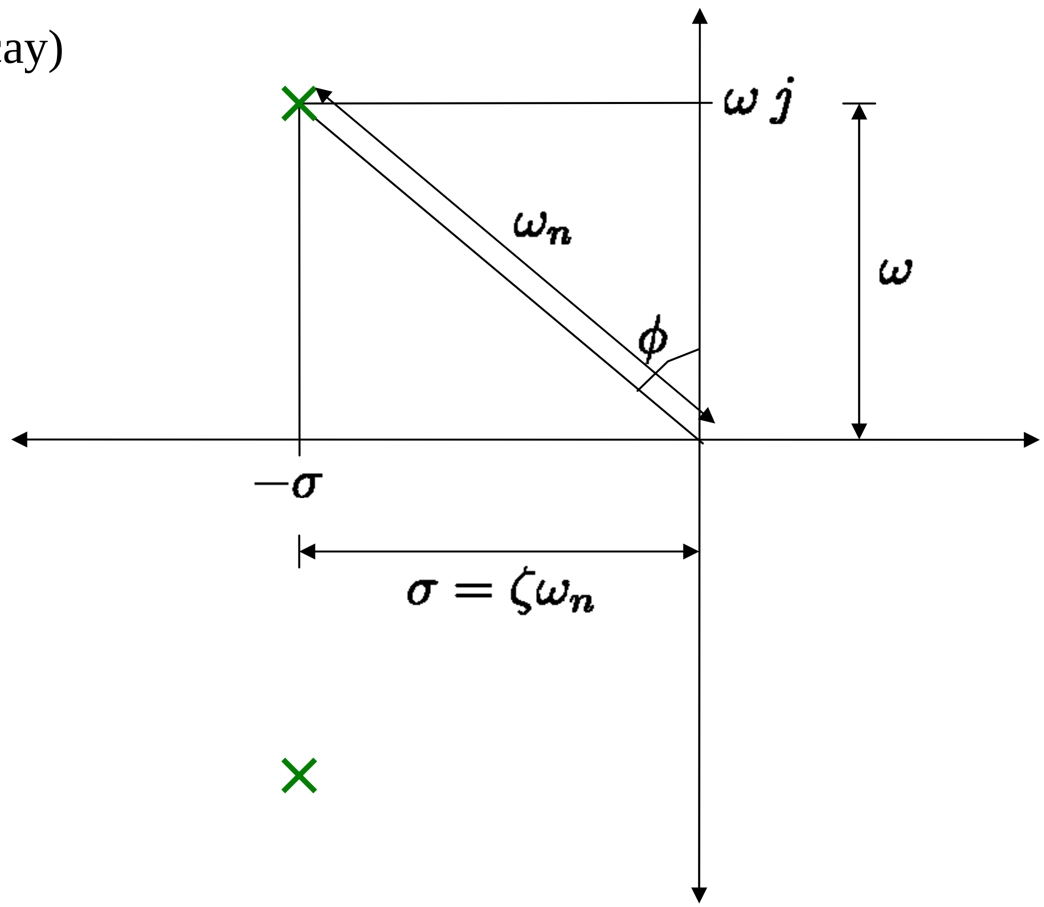
Polar representation :

ω_n … natural frequency

ζ … damping ratio



Polar vs. Cartesian representations.

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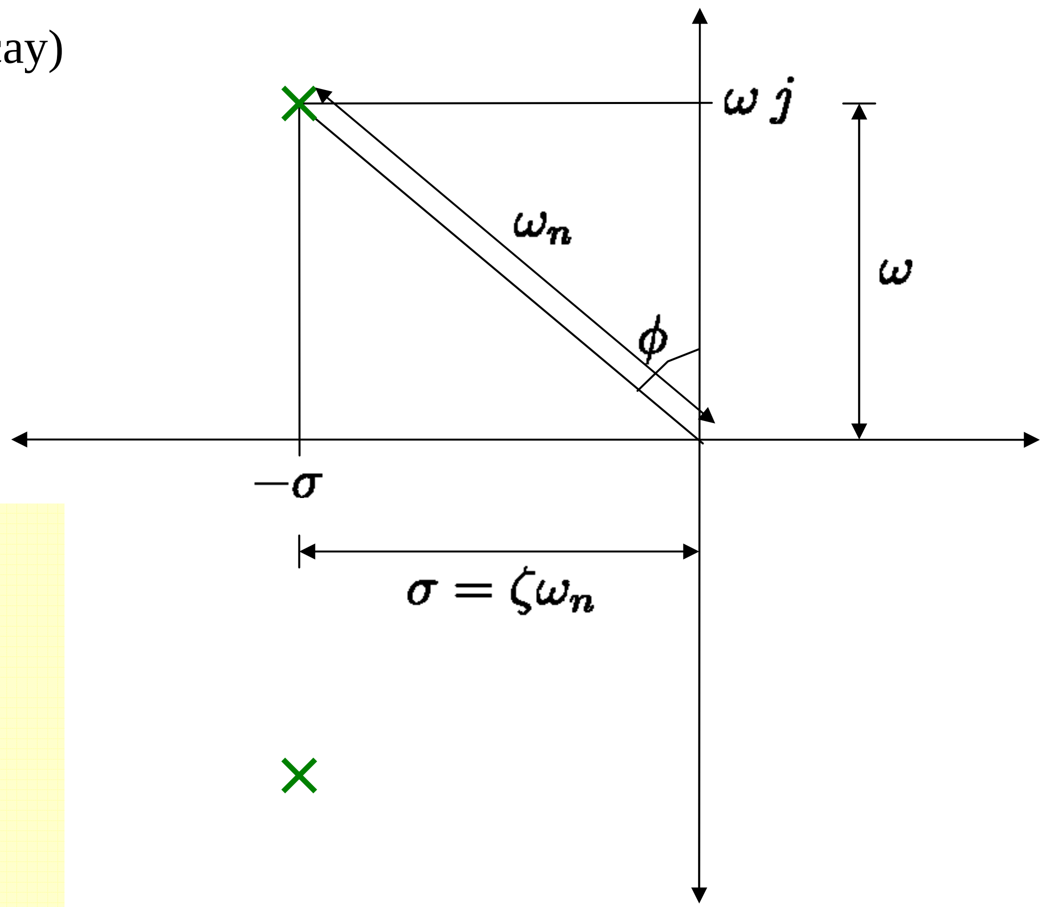
- ω … Imaginary part (frequency)
- σ … Real part (rate of decay)

Polar representation :

- ω_n … natural frequency
- ζ … damping ratio

$$\omega = |Im(p)|$$
$$\sigma = |Re(p)|$$
$$\omega_n = |p| = \sqrt{\sigma^2 + \omega^2}$$
$$\zeta = \sin(\phi) = \sigma / \omega_n$$
$$\omega = \omega_n \sqrt{1 - \zeta^2}$$



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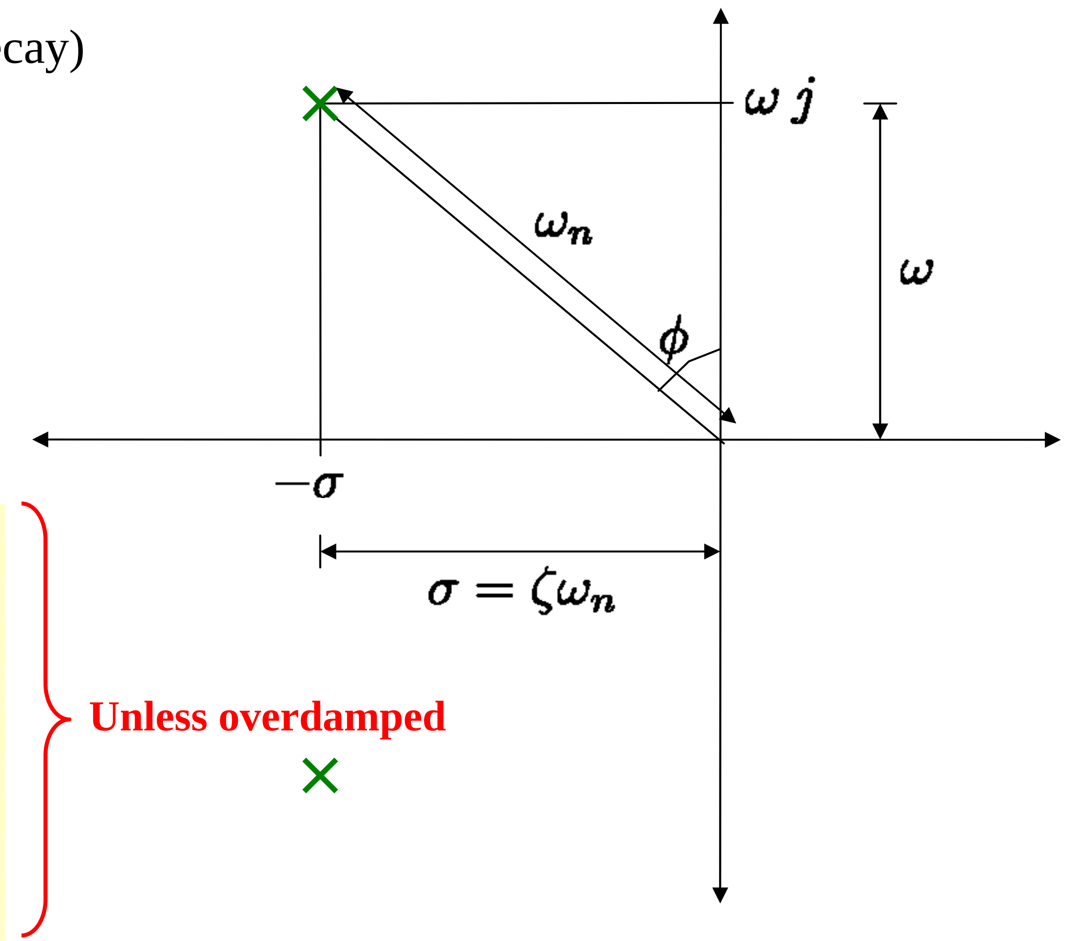

Polar vs. Cartesian representations.

- System transfer function :

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}}_{\textbf{Polar}} = \underbrace{\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}}_{\textbf{Cartesian}}$$

- Significance of the damping ratio :

$\zeta > 1$ … Overdamped

$\zeta = 1$ … Critically damped

$1 > \zeta > 0$ … Underdamped

$\zeta = 0$ … Undamped

Polar vs. Cartesian representations.

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Polar vs. Cartesian representations.

- System transfer function :

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$p_{1,2} = -\sigma \pm \omega\, j$$

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}}_{\text{Polar}} = \underbrace{\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}}_{\text{Cartesian}}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2\sigma s + (\sigma^2 + \omega^2)$$

$$\Rightarrow \sigma = \zeta\omega_n,\ \omega^2 = \omega_n^2(1 - \zeta^2)$$

- Significance of the damping ratio :

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Polar vs. Cartesian representations.

- System transfer function : $p_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

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$p_{1,2} = -\sigma \pm \omega\, j$

All 4 cases (Polar)

Unless overdamped (Cartesian)

- Significance of the damping ratio :

$\zeta > 1$ … Overdamped

$\zeta = 1$ … Critically damped

$1 > \zeta > 0$ … Underdamped

$\zeta = 0$ … Undamped

Underdamped second order system

- Underdamped $0 < \zeta < 1$

$$H(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} = \frac{\sigma^2 + w^2}{(s+\sigma)^2 + w^2}$$

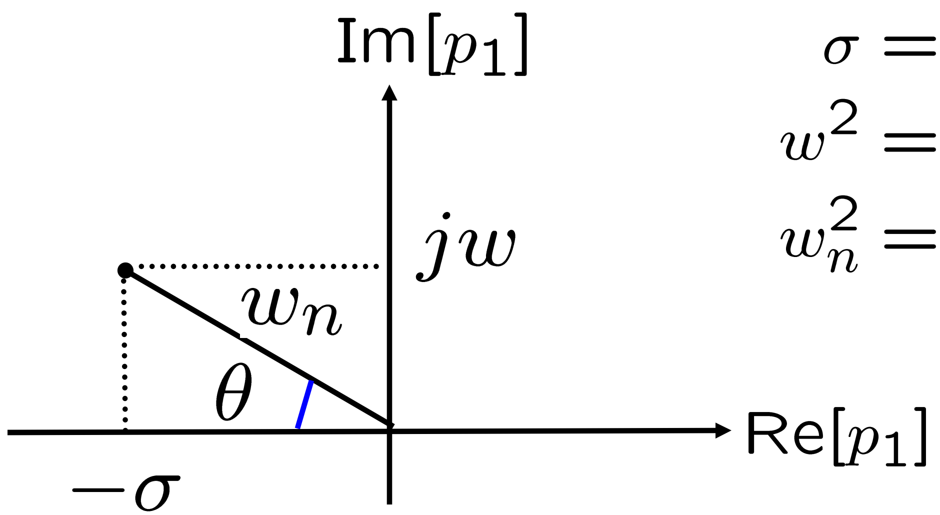
- Two complex poles:

$$p_1 = -\sigma + jw;\ p_2 = -\sigma - jw$$

Underdamped second order system

$$p_1 = -\sigma + jw = w_n e^{j(\pi-\theta)} = -w_n e^{-j\theta}$$

$$\theta = \tan^{-1}\left(\frac{w}{\sigma}\right) = \tan^{-1}\left((w_n\sqrt{1-\zeta^2})/\ (\zeta w_n)\right)$$
$$= \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$$



$$\sigma = \zeta w_n$$
$$w^2 = w_n^2(1-\zeta^2)$$
$$w_n^2 = \sigma^2 + w^2$$

Impulse response of the second order system

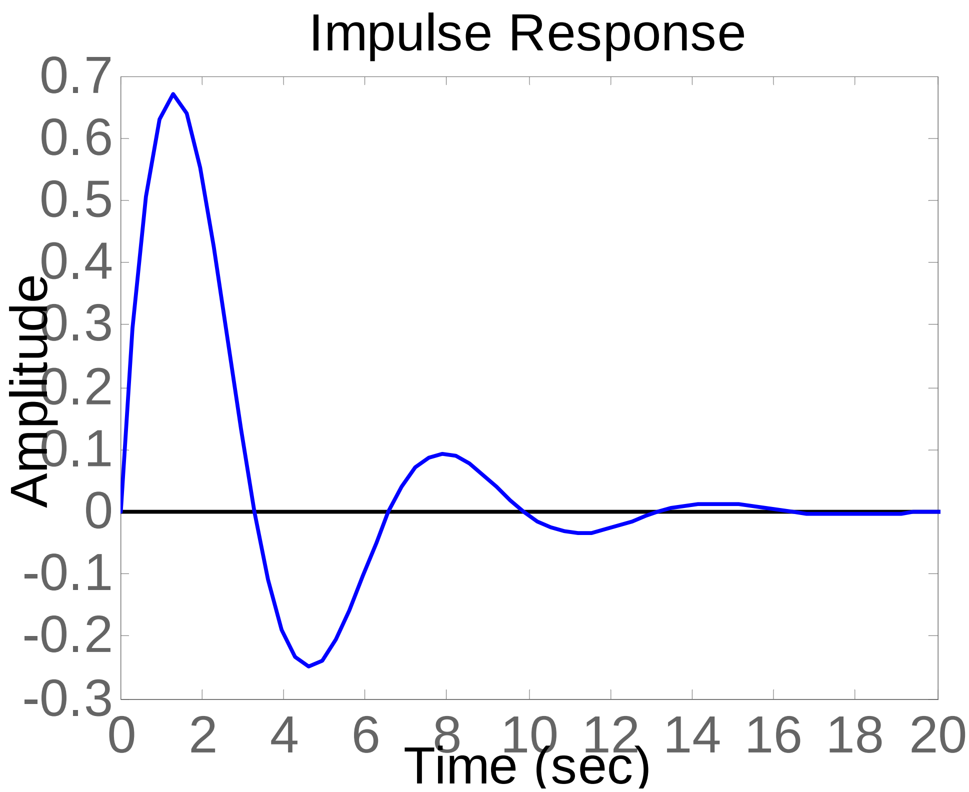
$$
\begin{aligned}
h(t) &= \mathcal{L}^{-1}\left[\frac{\sigma^2+w^2}{(s+\sigma)^2+w^2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{\sigma^2+w^2}{w}\frac{w}{(s+\sigma)^2+w^2}\right] \\
&= \frac{\sigma^2+w^2}{w}e^{-\sigma t}\sin(wt) \\
&= \frac{w_n^2}{w_n\sqrt{1-\zeta^2}}e^{-\zeta w_n t}\sin(w_n\sqrt{1-\zeta^2}t)
\end{aligned}
$$

$$
h(t) = \frac{w_n}{\sqrt{1-\zeta^2}}e^{-\zeta w_n t}\sin(w_n\sqrt{1-\zeta^2}t)1(t)
$$

Matlab Simulation

$$G(s) = \frac{1}{s^2 + 0.6s + 1}$$

- zeta = 0.3; wn=1;
- G=tf([wn],[1 2*zeta*wn wn^2]);
- impulse(G)



Unit step response of undamped systems

- Unit step response :

$$y_{step}(t) = \mathcal{L}^{-1}[H(s)/s] = \mathcal{L}^{-1}\left[\frac{\sigma^2 + w^2}{((s+\sigma)^2 + w^2)s}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s+a)}{(s+\sigma)^2 + w^2}\right]$$

$$a = 2\sigma = 2\zeta w_n$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s+\sigma)}{(s+\sigma)^2 + w^2} - \frac{\sigma}{w}\frac{w}{(s+\sigma)^2 + w^2}\right]$$

$$= \left(1 - e^{-\sigma t}\left[\cos(wt) + \frac{\sigma}{w}\sin(wt)\right]\right) 1(t)$$

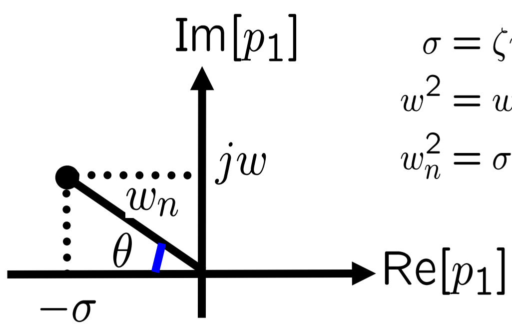
- DC gain :

$$y_{ss} = \lim_{t\to\infty} y_{step}(t) = \lim_{s\to 0}[H(s)/s]s = 1$$

Unit step response of undamped system

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\begin{aligned}
&\Rightarrow \left[\cos(wt) + \frac{\sigma}{w}\sin(wt)\right] \\
&= \frac{w_n}{w}\left[\frac{w}{w_n}\cos(wt) + \frac{\sigma}{w_n}\sin(wt)\right] \\
&= \frac{w_n}{w}[\sin(\theta)\cos(wt) + \cos(\theta)\sin(wt)] \\
&= \frac{w_n}{w}\sin(wt+\theta)
\end{aligned}$$



$$\sigma = \zeta w_n$$
$$w^2 = w_n^2(1-\zeta^2)$$
$$w_n^2 = \sigma^2 + w^2$$

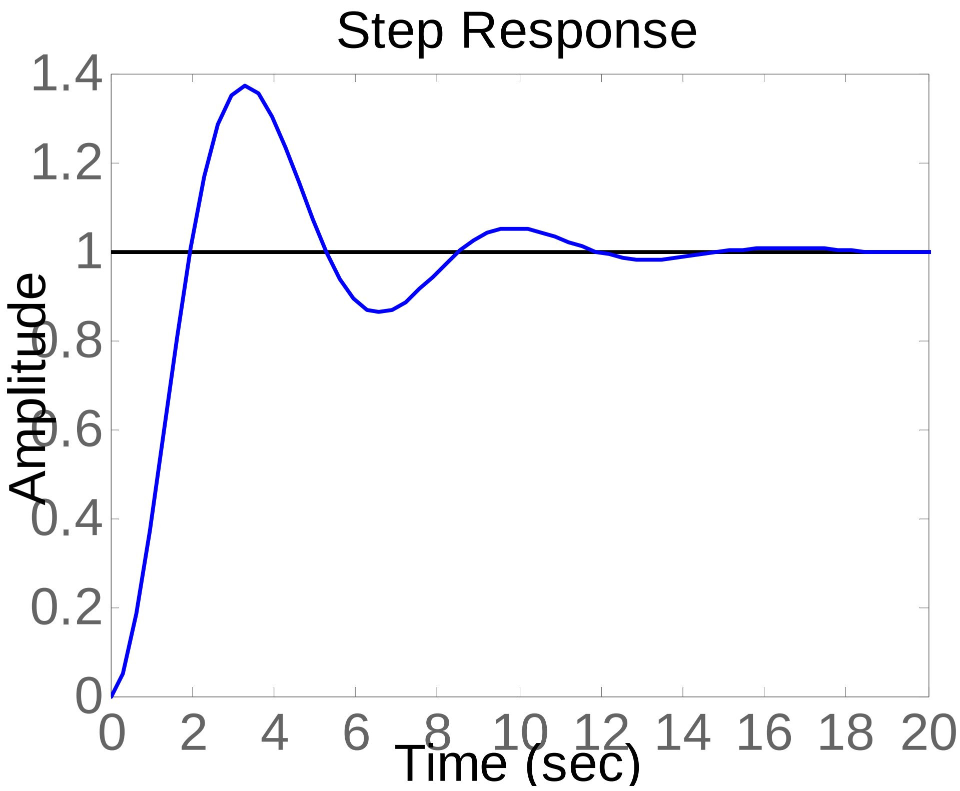
$$\theta = \tan^{-1}\left(\frac{w}{\sigma}\right) = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$$

$$\begin{aligned}
y_{step}(t) &= \left(1 - e^{-\sigma t}\frac{w_n}{w}\sin(wt+\theta)\right)1(t) \\
&= \left(1 - \frac{e^{-\zeta w_n t}}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\,w_n t + \theta)\right)1(t)
\end{aligned}$$

Matlab Simulation

$$G(s) = \frac{1}{s^2 + 0.6s + 1}$$

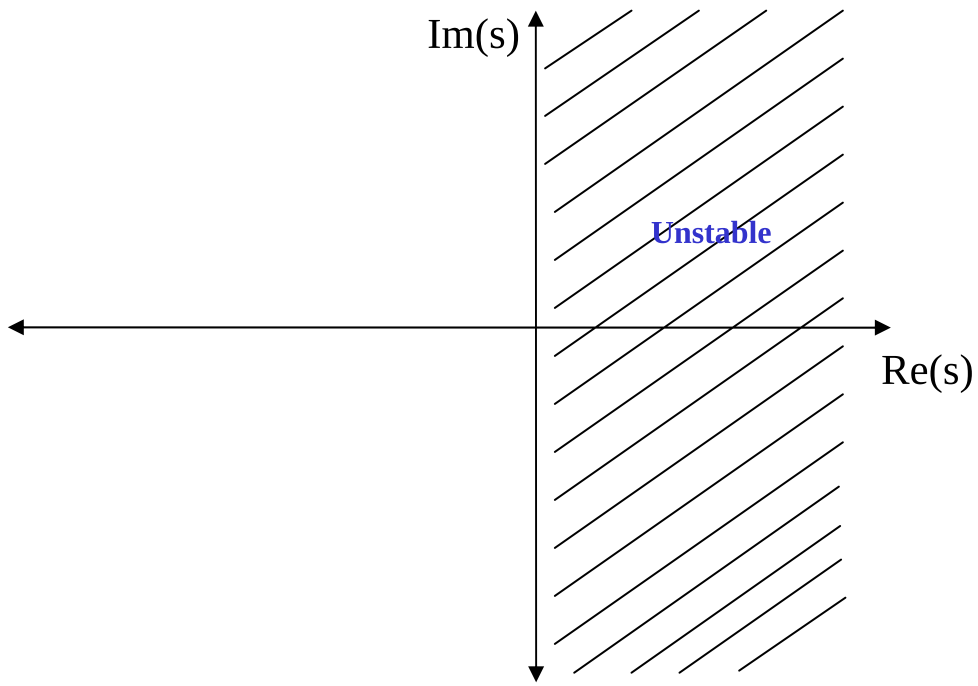
- zeta = 0.3; wn=1; G=tf([wn],[1 2*zeta*wn wn^2]);
- step(G)



Second order system response.

Stable 2nd order system:

- 2 distinct real poles
- A pair of repeated real poles
- A pair of complex poles



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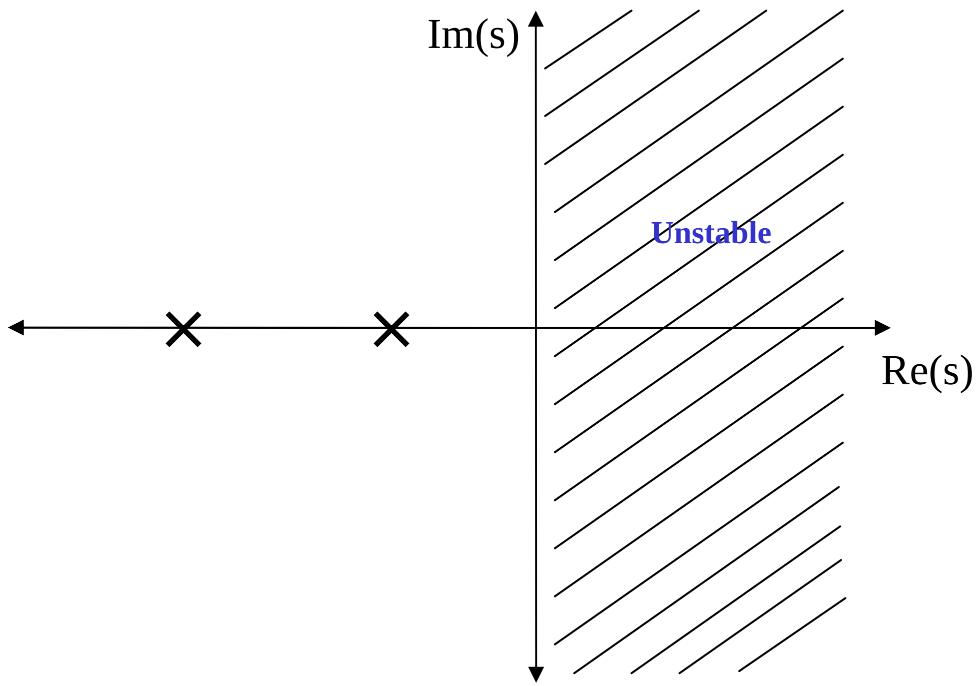

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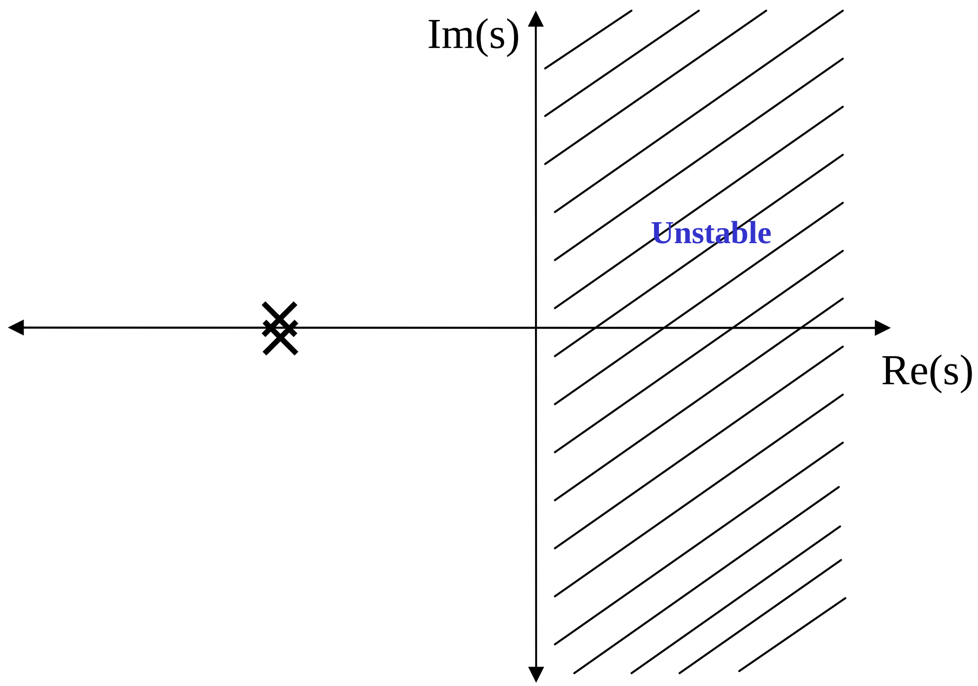
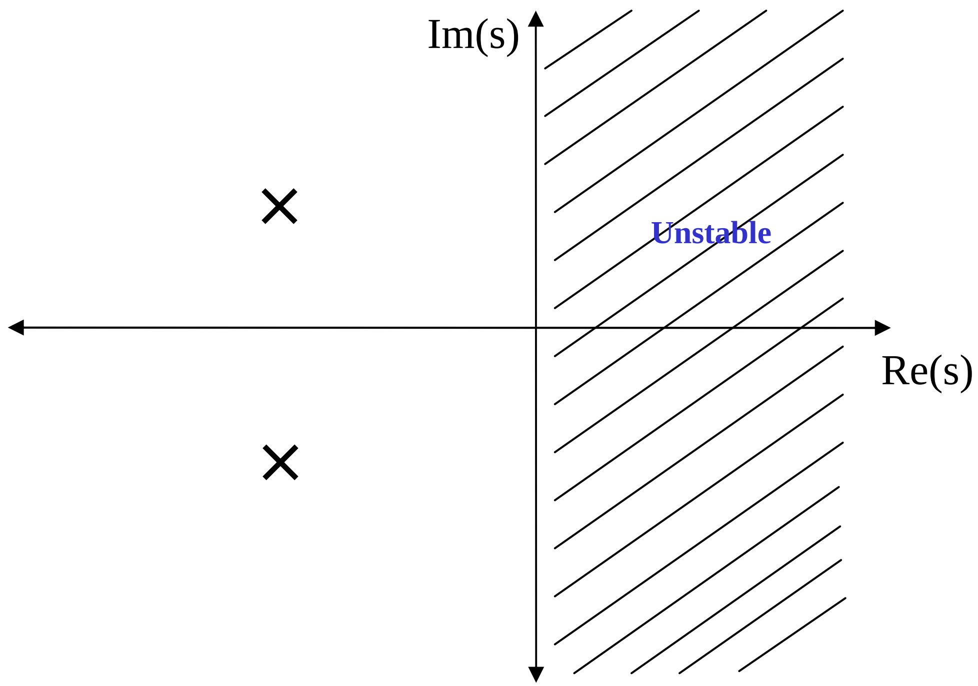


Second order system response.

Stable 2nd order system:

- 2 distinct real poles
- A pair of repeated real poles
- A pair of complex poles → negative real part / zero real part



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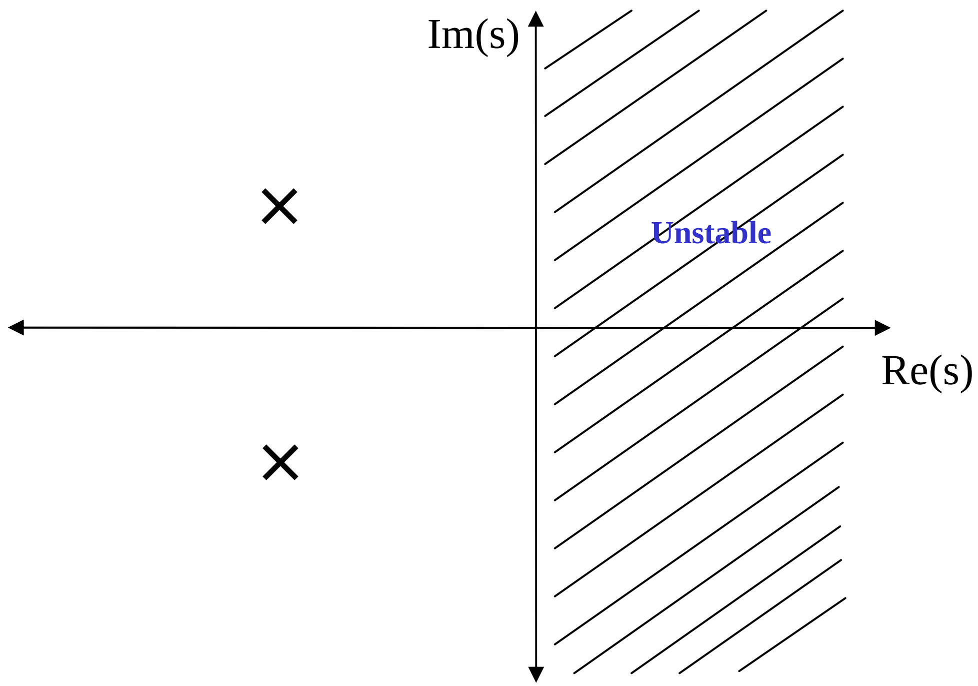
- A pair of complex poles
  - → negative real part
  - → zero real part



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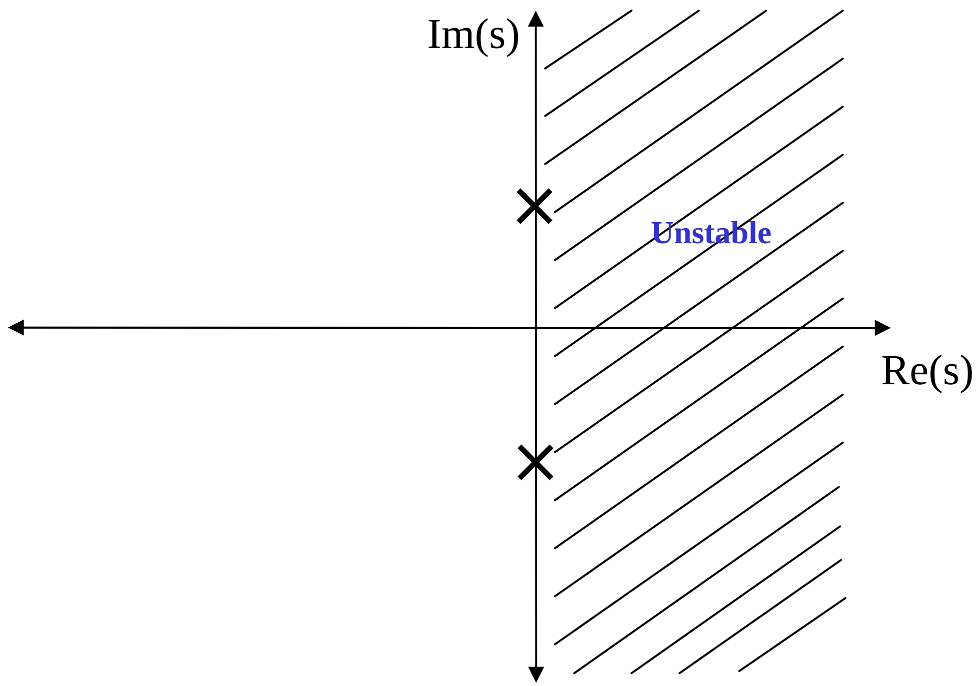
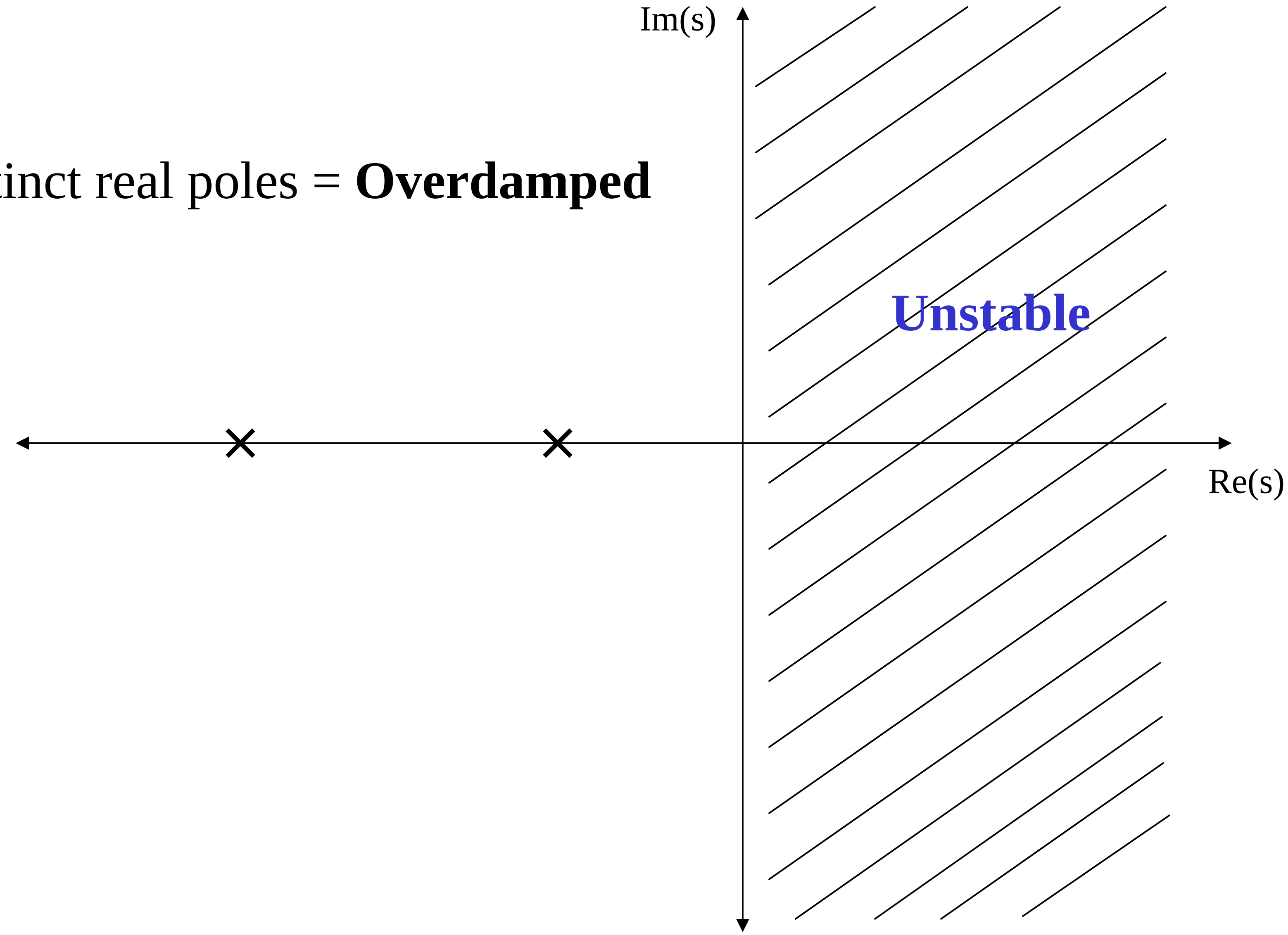
negative real part

zero real part



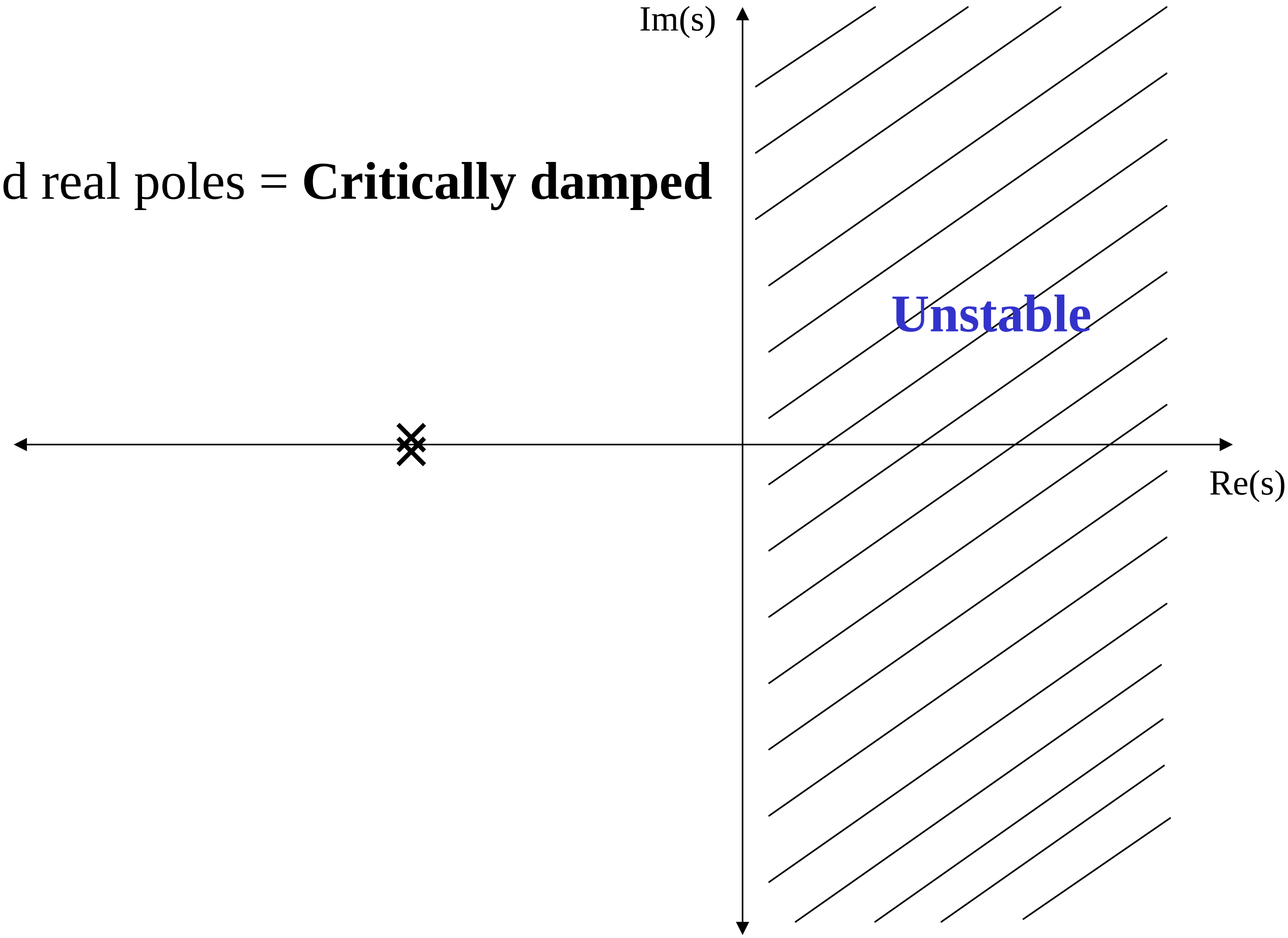
Second order system response.

2 distinct real poles = **Overdamped**



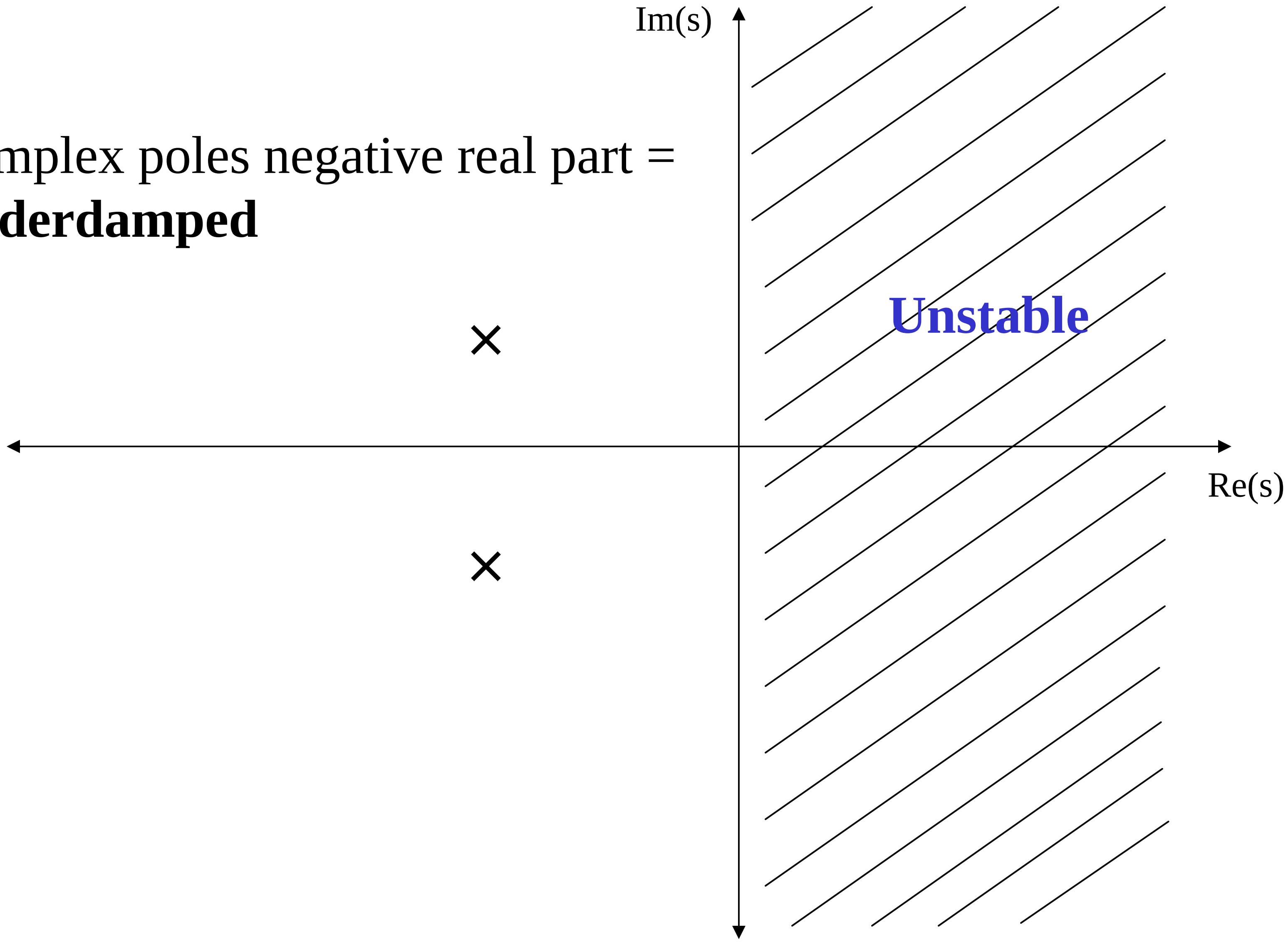
Second order system response.

Repeated real poles = **Critically damped**



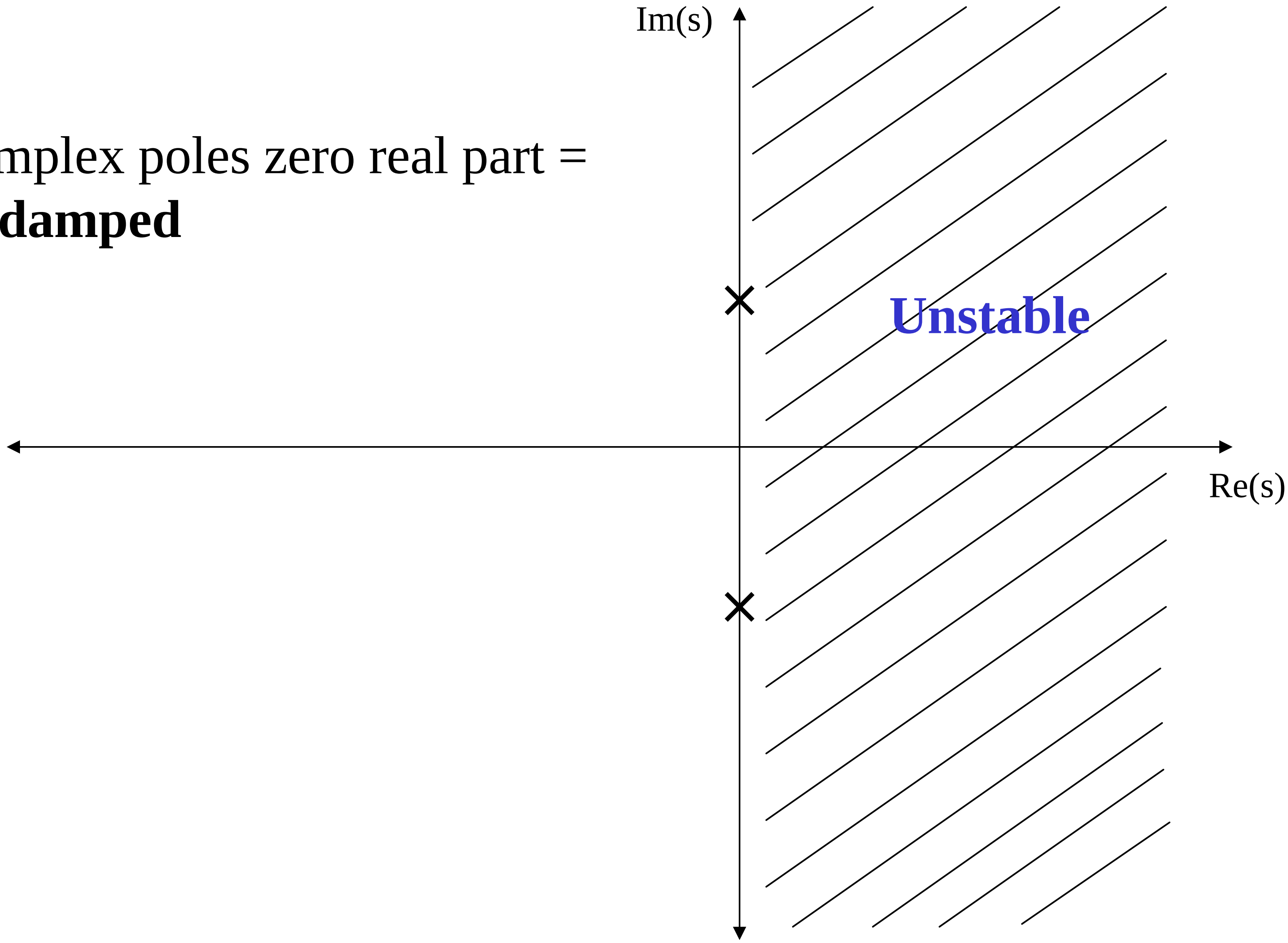
Second order system response.

Complex poles negative real part =
Underdamped

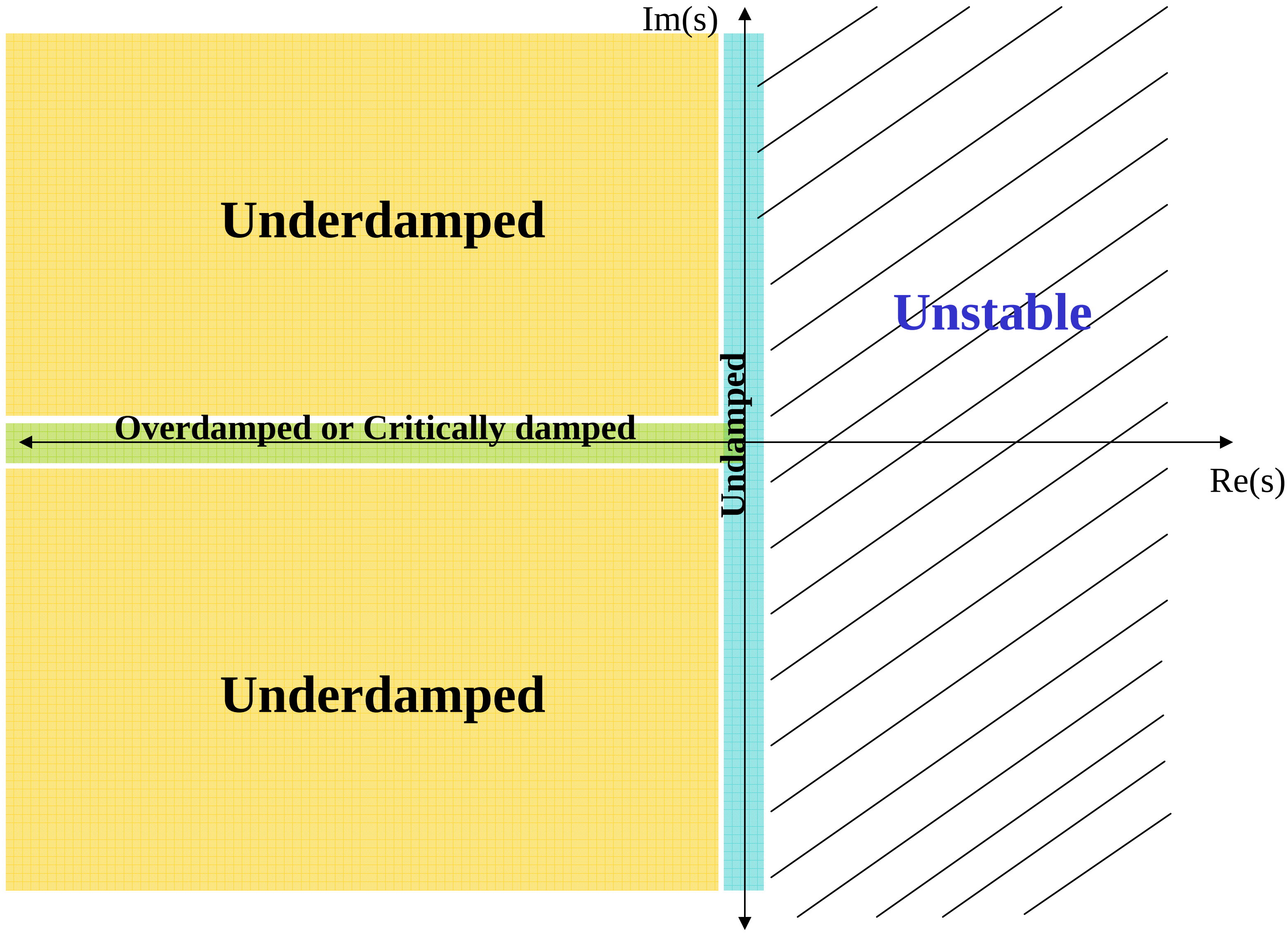


Second order system response.

Complex poles zero real part =
Undamped

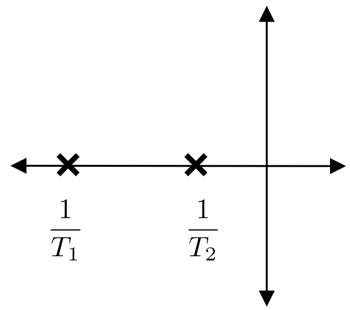


Second order system response.



Overdamped system response

- System transfer function : $H(s) = \dfrac{K}{(T_1 s + 1)(T_2 s + 1)}$



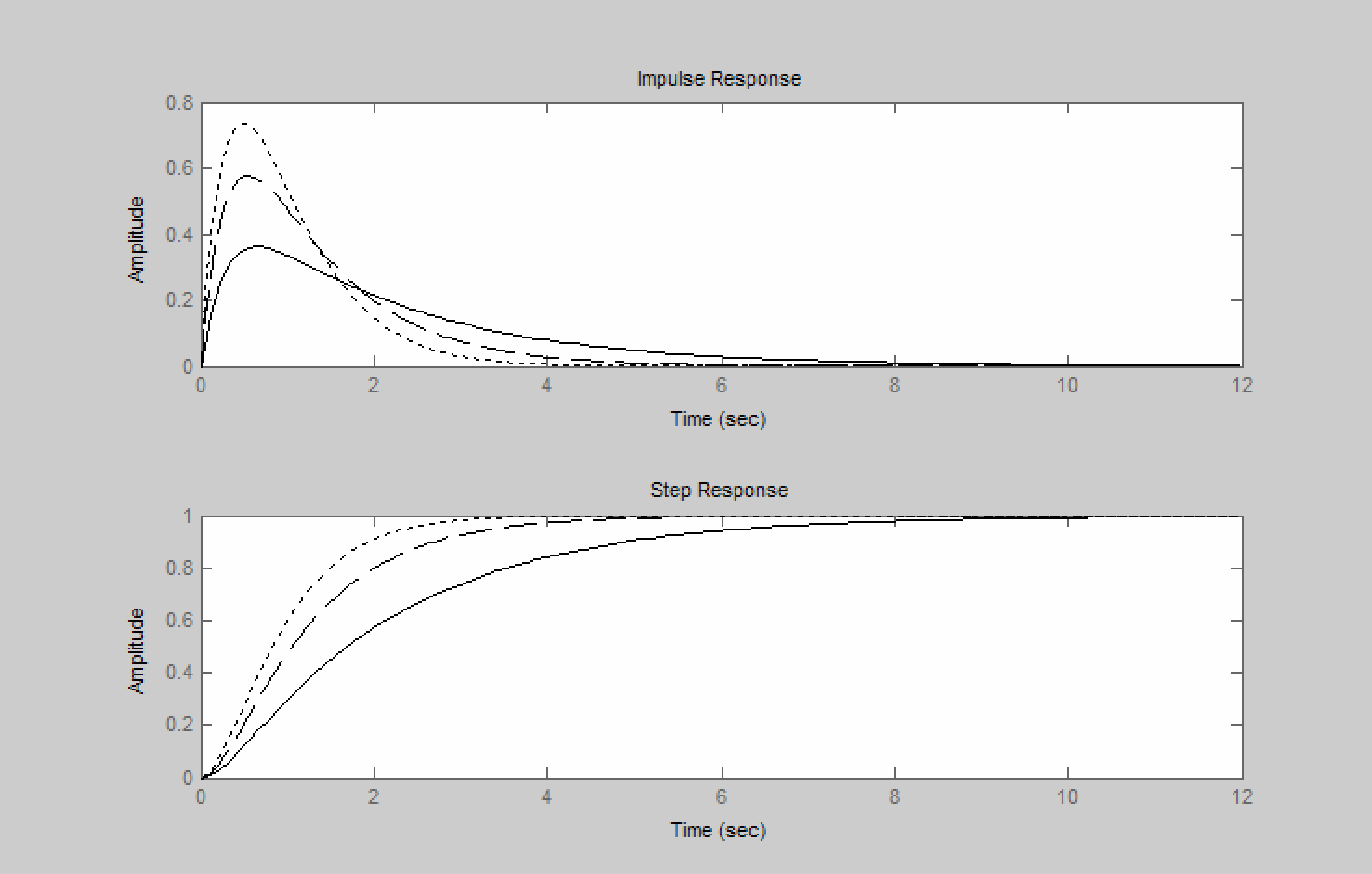
- Impulse response :

$$h(t) \;=\; \mathcal{L}^{-1}[H(s)] \;=\; \frac{K}{T_2 - T_1}\left(e^{-t/T_2} - e^{-t/T_1}\right) 1(t)$$

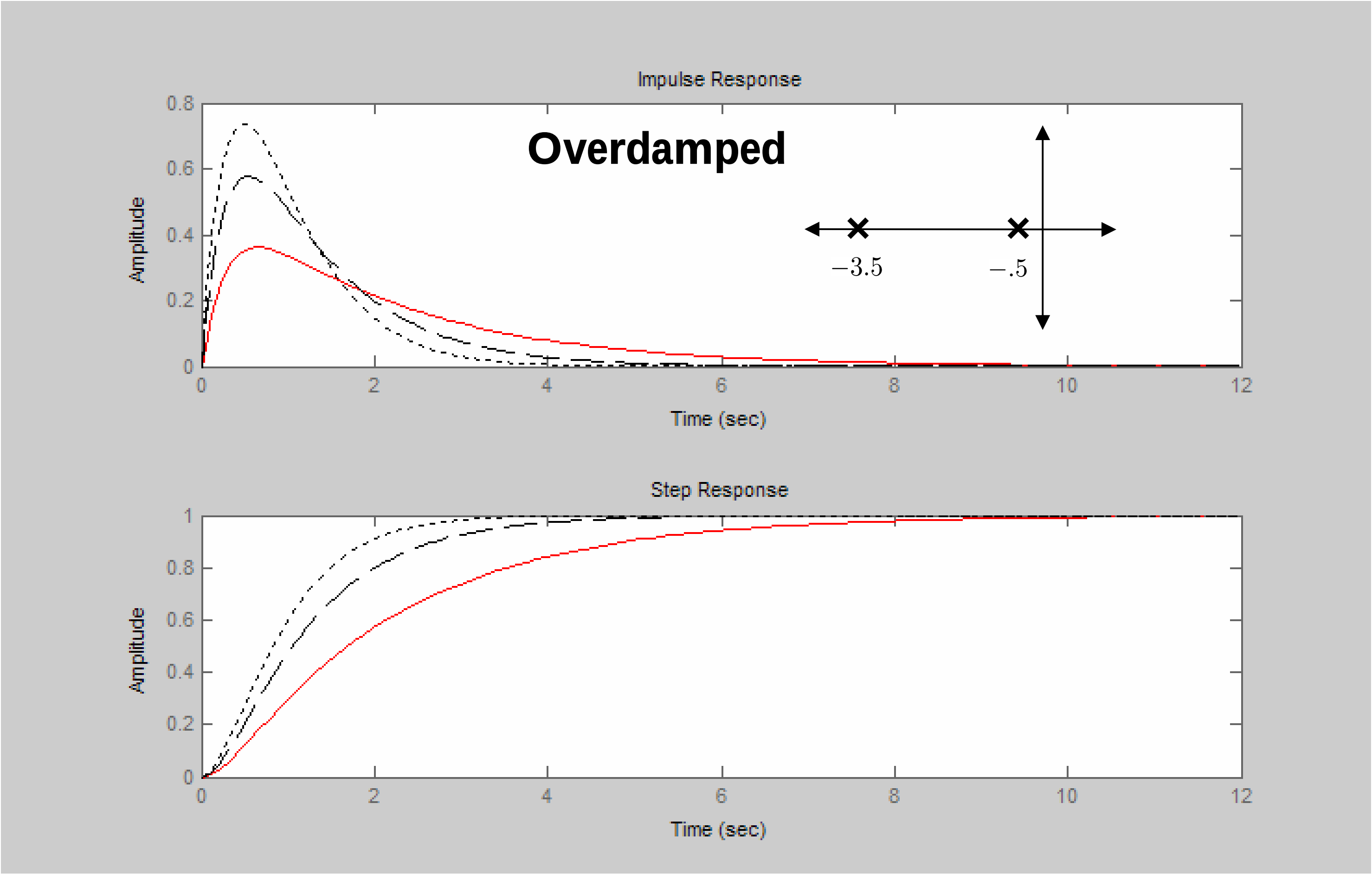
- Step response :

$$y_{step}(t) \;=\; \mathcal{L}^{-1}[H(s)/s] \;=\; K\left(1 + \frac{T_1}{T_2 - T_1} e^{-t/T_2} - \frac{T_2}{T_2 - T_1} e^{-t/T_1}\right) 1(t)$$

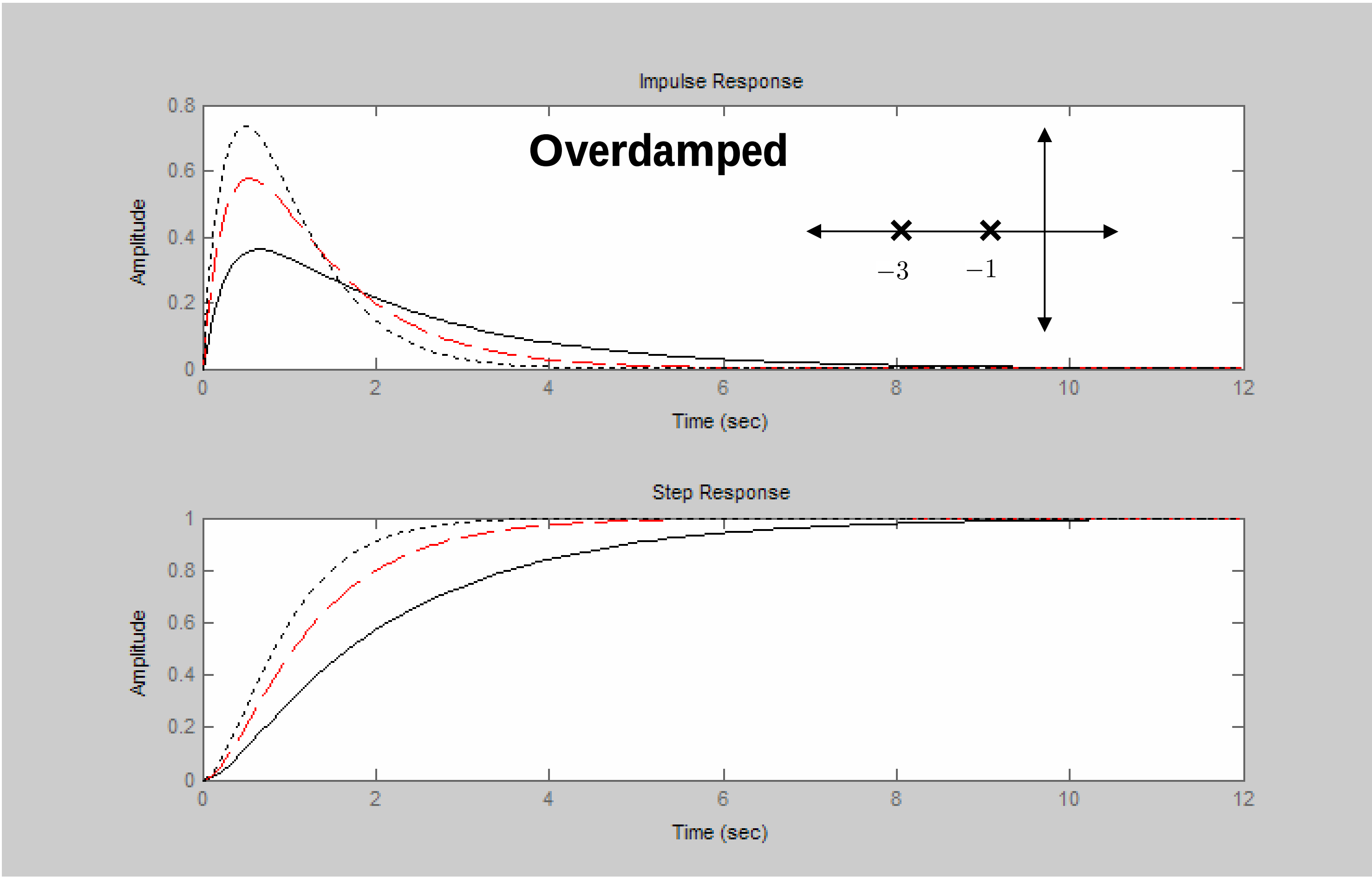
Overdamped and critically damped system response.



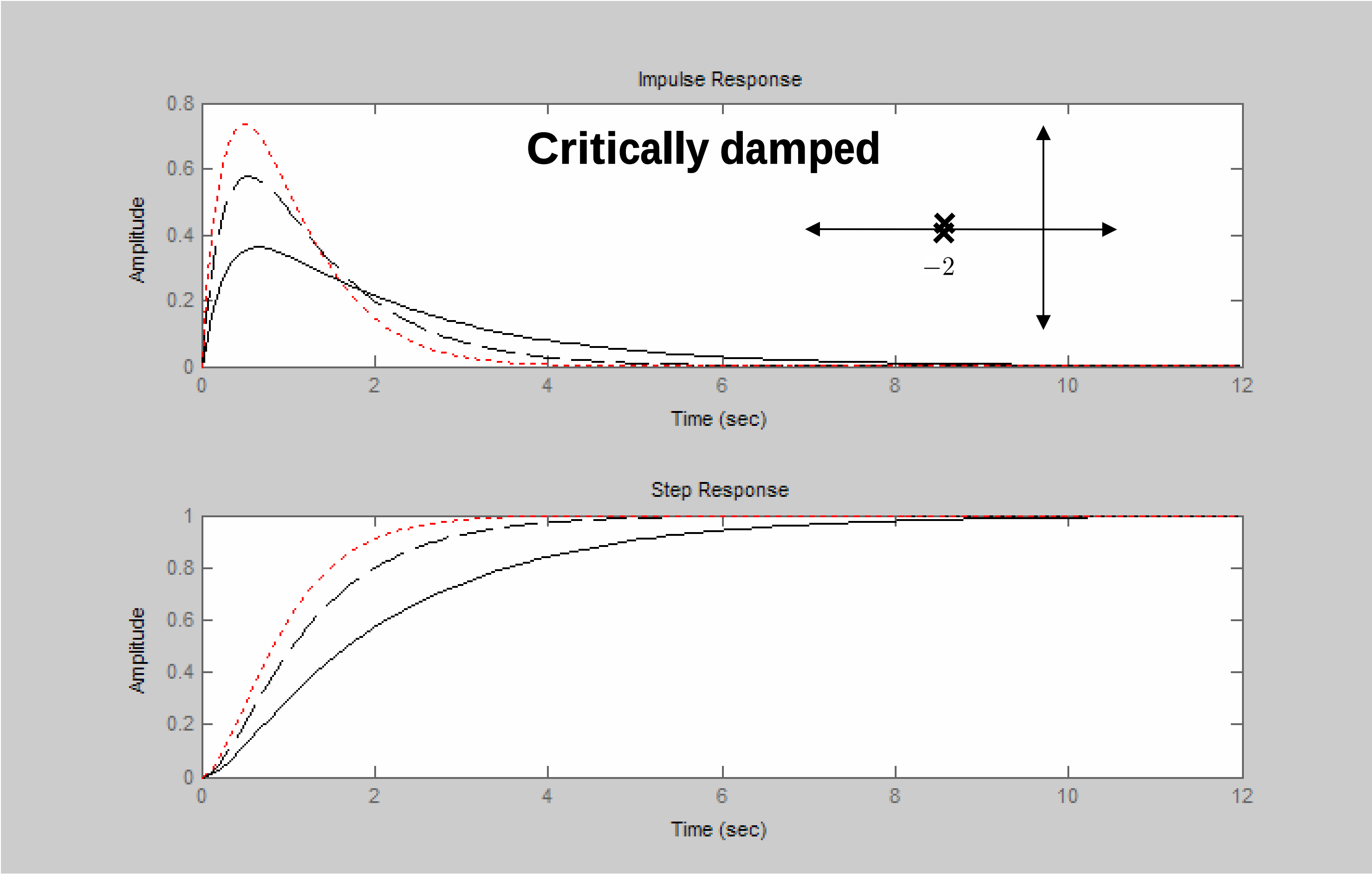
Overdamped and critically damped system response.



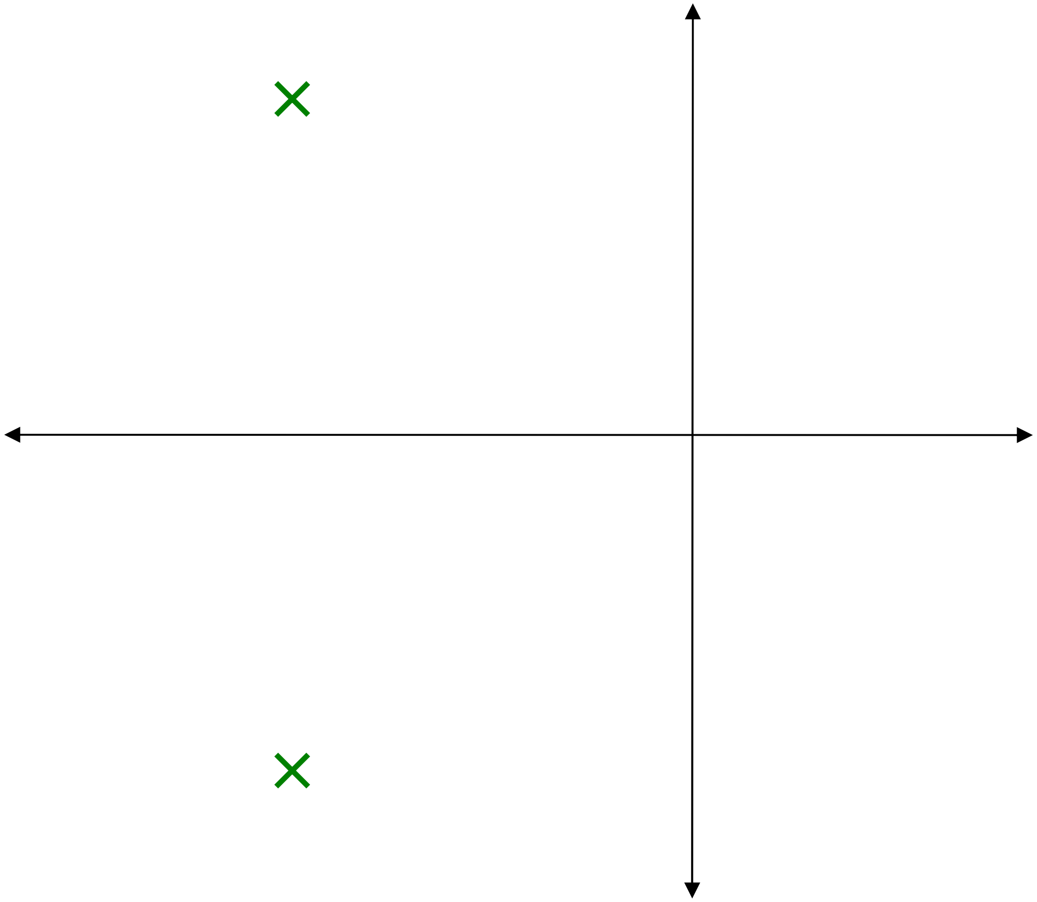
Overdamped and critically damped system response.



Overdamped and critically damped system response.



Polar vs. Cartesian representations.



Polar vs. Cartesian representations.

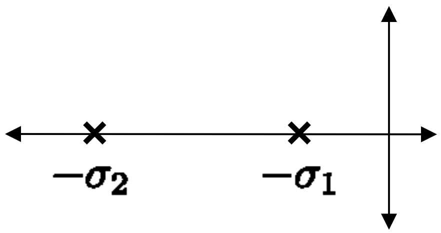
- System transfer function :

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}}_{\text{Polar}} = \underbrace{\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}}_{\text{Cartesian}}$$

$$p_{1,2} = -\sigma \pm \omega\, j$$

All 4 cases (Polar) — **Unless overdamped** (Cartesian)



$$H(s) = \frac{\sigma_1\sigma_2}{(s+\sigma_1)(s+\sigma_2)}$$

… Cartesian overdamped

$$p_1 = -\sigma_1\ ,\ p_2 = -\sigma_2$$

- Significance of the damping ratio :

$\zeta > 1$ … Overdamped

$\zeta = 1$ … Critically damped

$1 > \zeta > 0$ … Underdamped

$\zeta = 0$ … Undamped

Polar vs. Cartesian representations.

- System transfer function :

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

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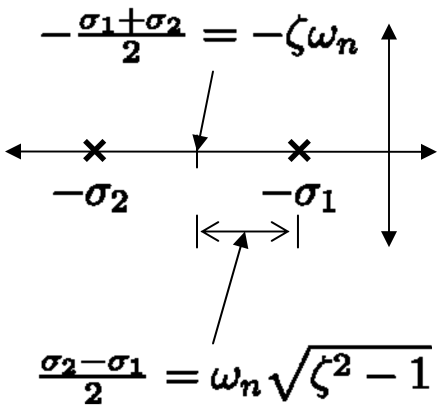

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}}_{\text{Polar}} = \underbrace{\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}}_{\text{Cartesian}}$$

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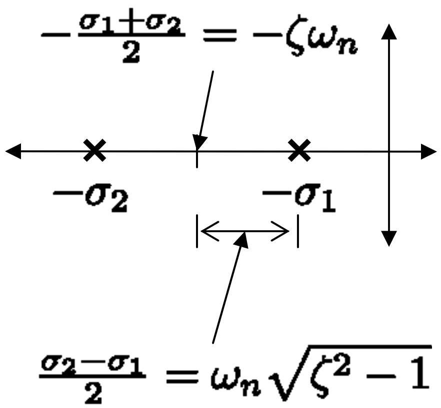

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$p_1 = -\sigma_1\ ,\ p_2 = -\sigma_2$

- Significance of the damping ratio :

$\zeta > 1$	… Overdamped
$\zeta = 1$	… Critically damped
$1 > \zeta > 0$	… Underdamped
$\zeta = 0$	… Undamped

Overdamped case:

$$-\sigma_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\sigma_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$\omega_n = \sqrt{\sigma_1\sigma_2}$$

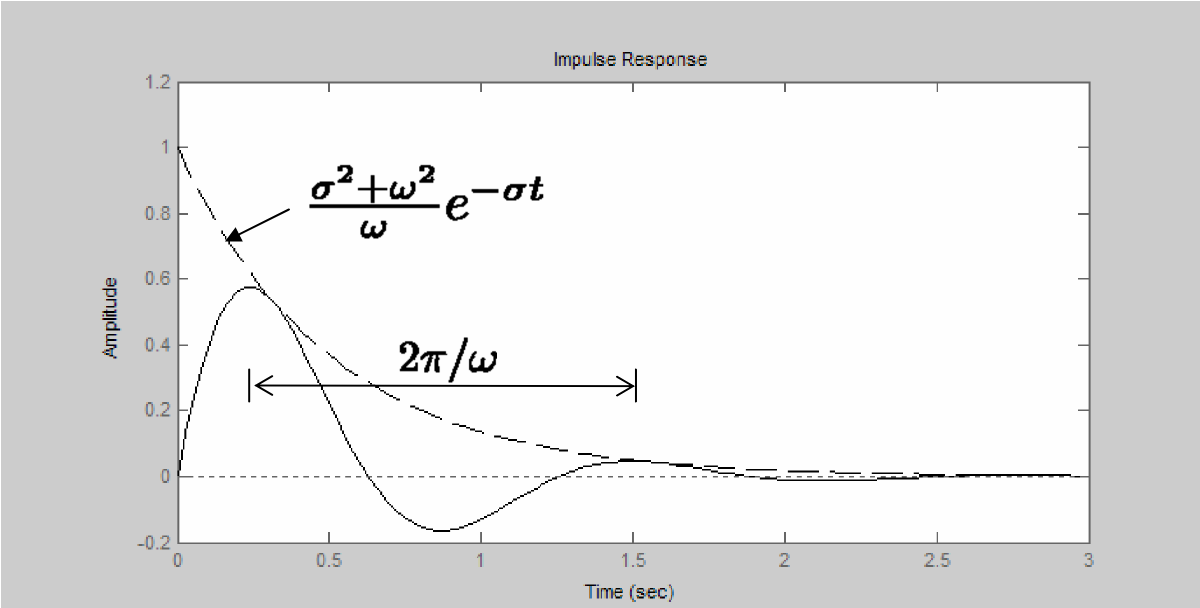
$$\zeta = \frac{\sigma_1 + \sigma_2}{2\sqrt{\sigma_1\sigma_2}}$$

Second order impulse response – Underdamped and Undamped

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}}_{\text{Polar}} = \underbrace{\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}}_{\text{Cartesian}}$$

- Impulse response :

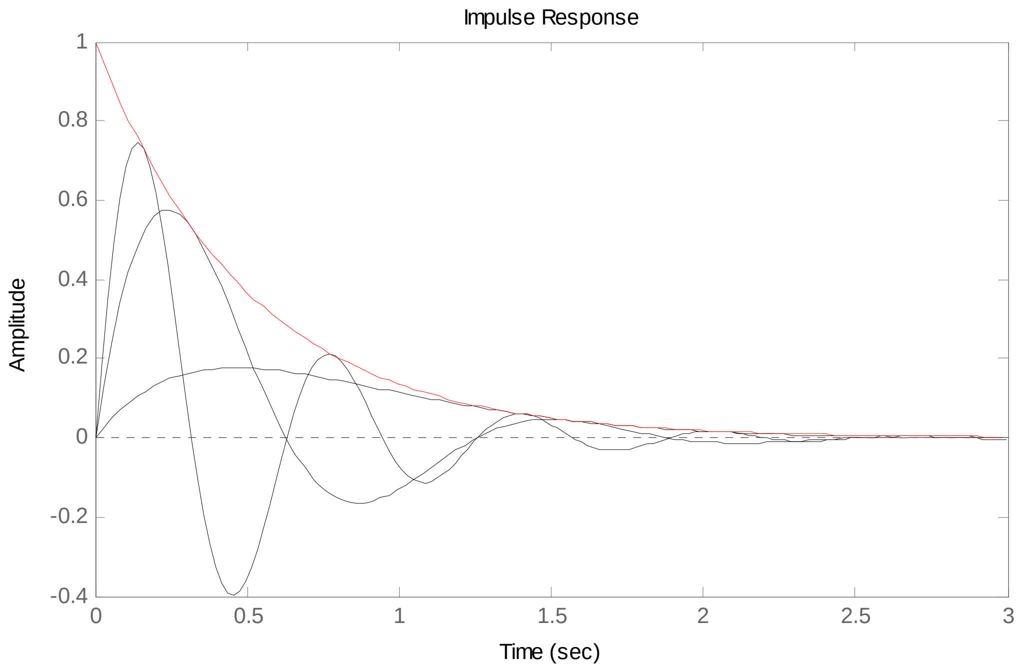
$$h(t) = \mathcal{L}^{-1}\left[\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}\right] = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$



Second order impulse response – Underdamped and Undamped

Increasing ω / Fixed σ

$$H(s) = \frac{\sigma^2+\omega^2}{(s+\sigma)^2+\omega^2}$$

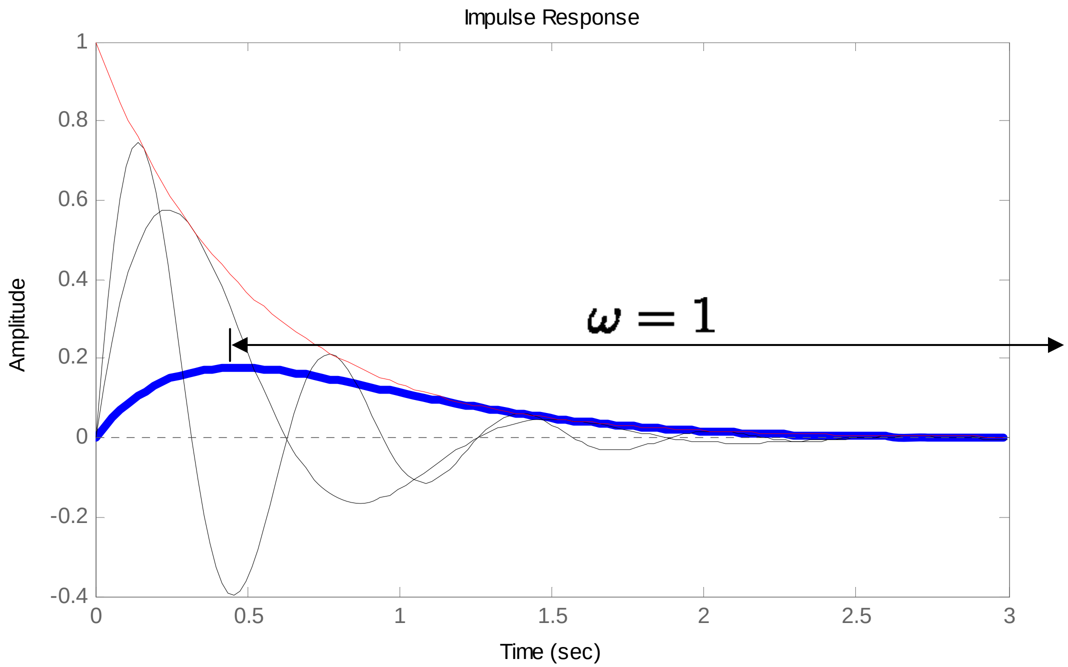


$$h(t) = \frac{\sigma^2+\omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing ω / Fixed σ

$$H(s) = \frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}$$



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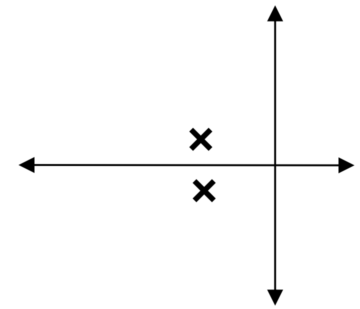

Second order impulse response – Underdamped and Undamped

Increasing ω / Fixed σ

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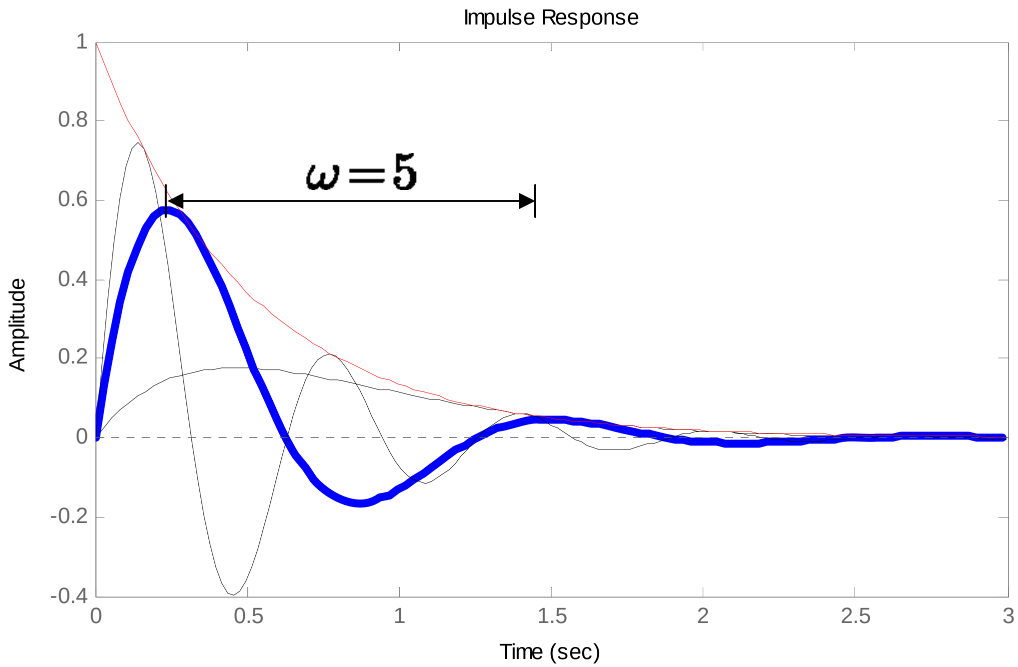

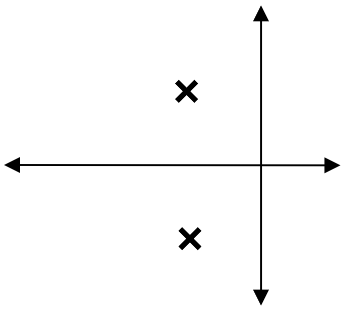

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Second order impulse response – Underdamped and Undamped

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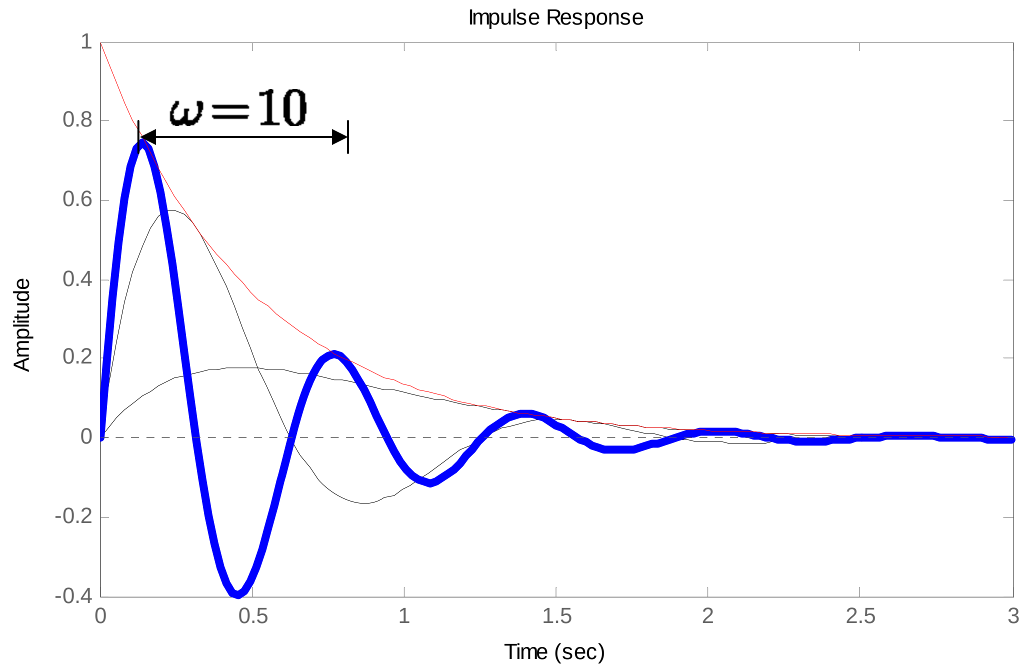

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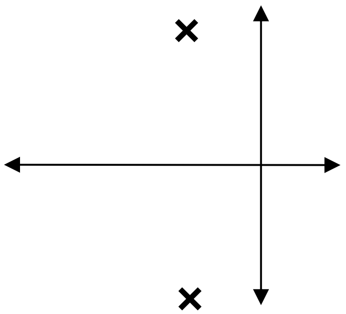

Second order impulse response – Underdamped and Undamped

Increasing σ / Fixed ω

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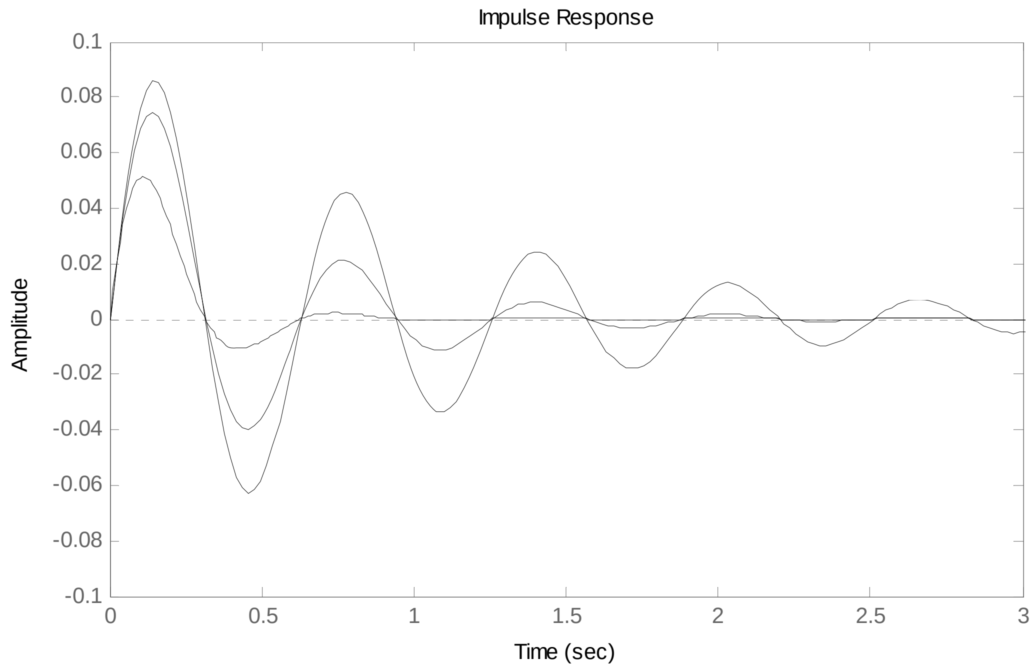

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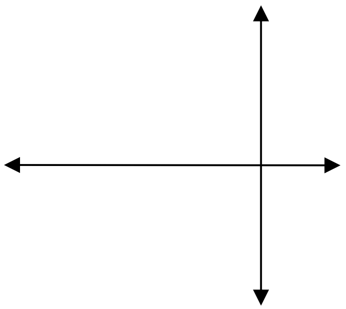

Second order impulse response – Underdamped and Undamped

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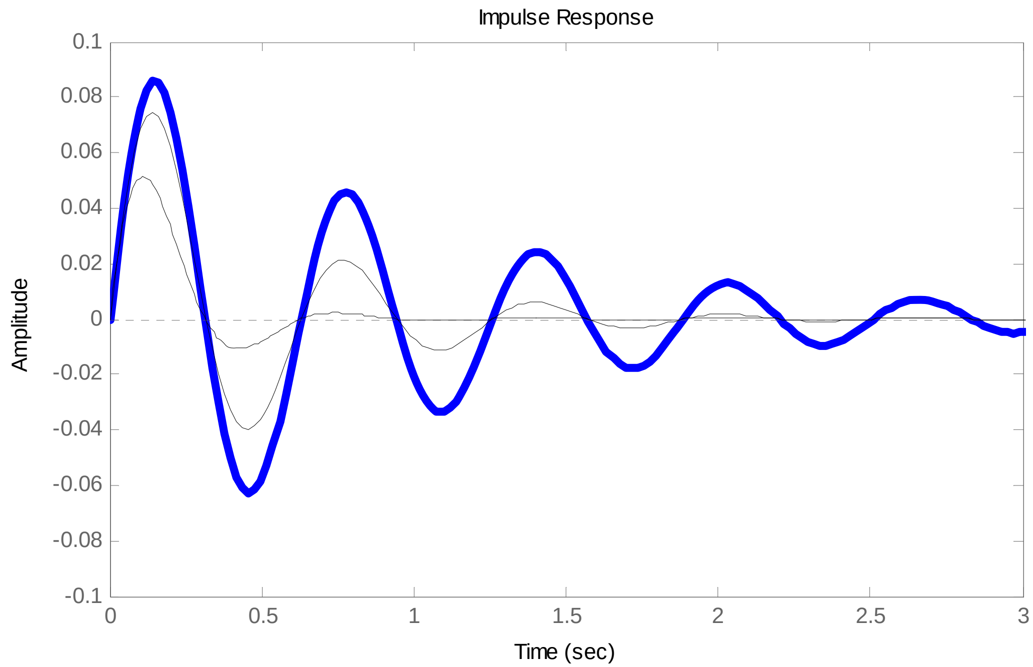

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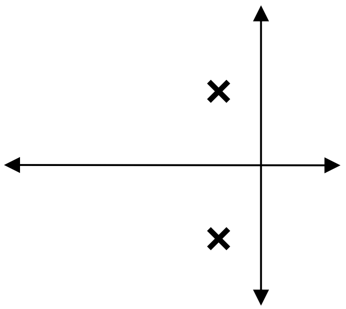

Second order impulse response – Underdamped and Undamped

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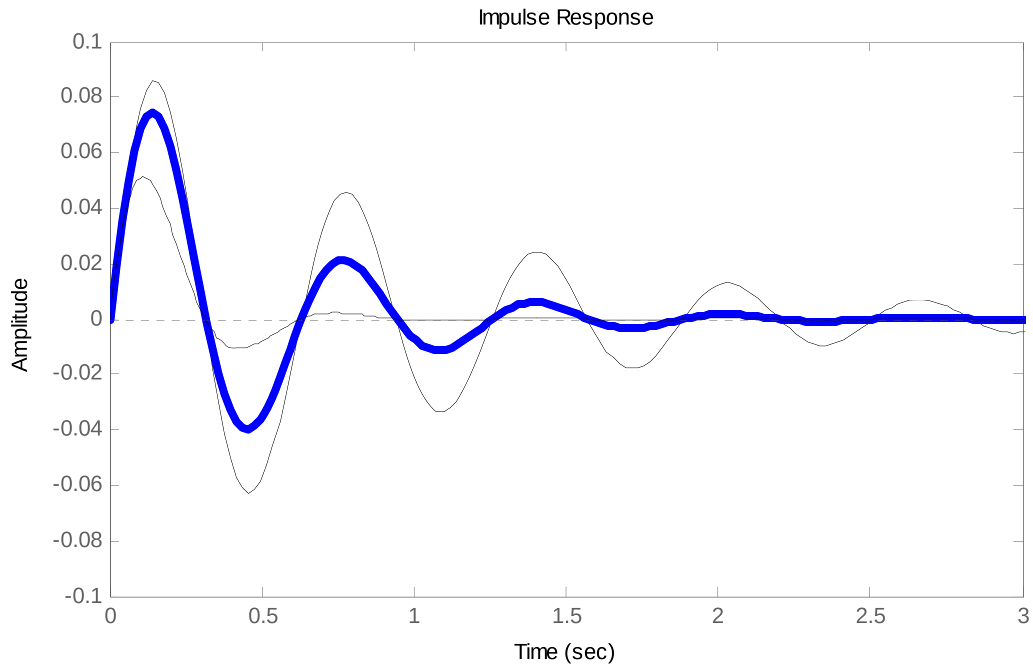

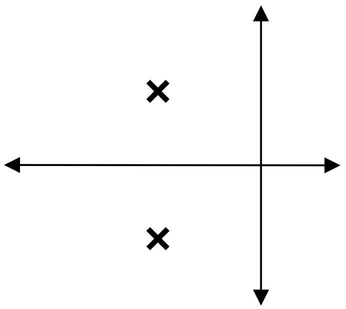

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Second order impulse response – Underdamped and Undamped

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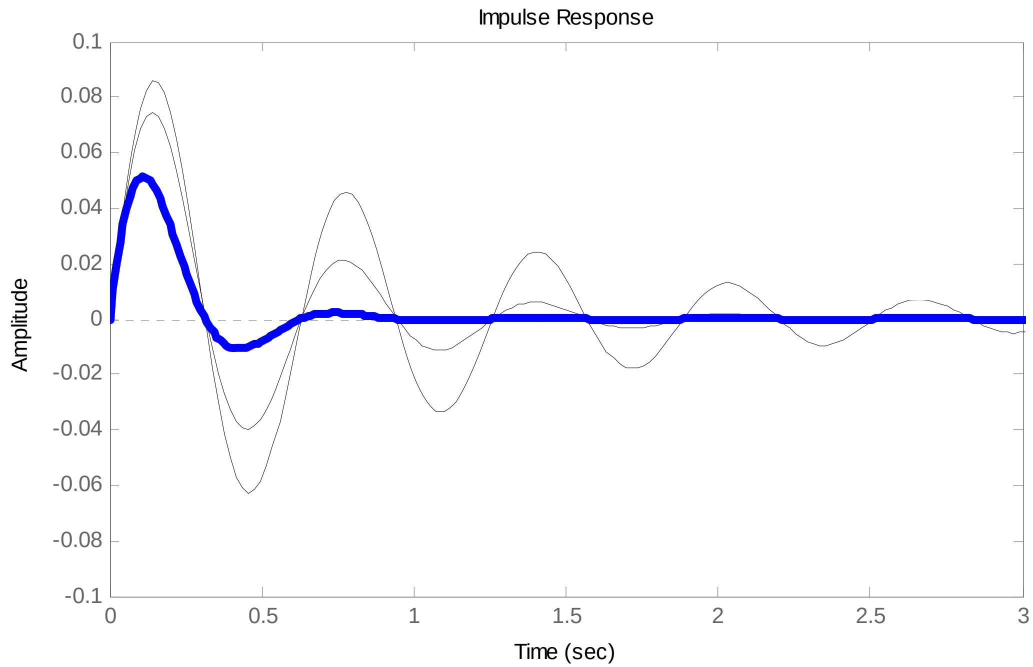
$$H(s) = \frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}$$



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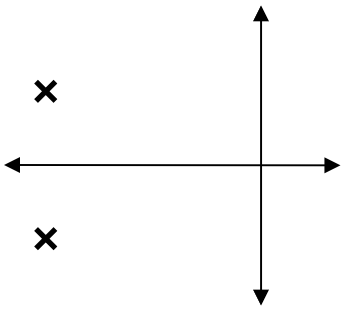
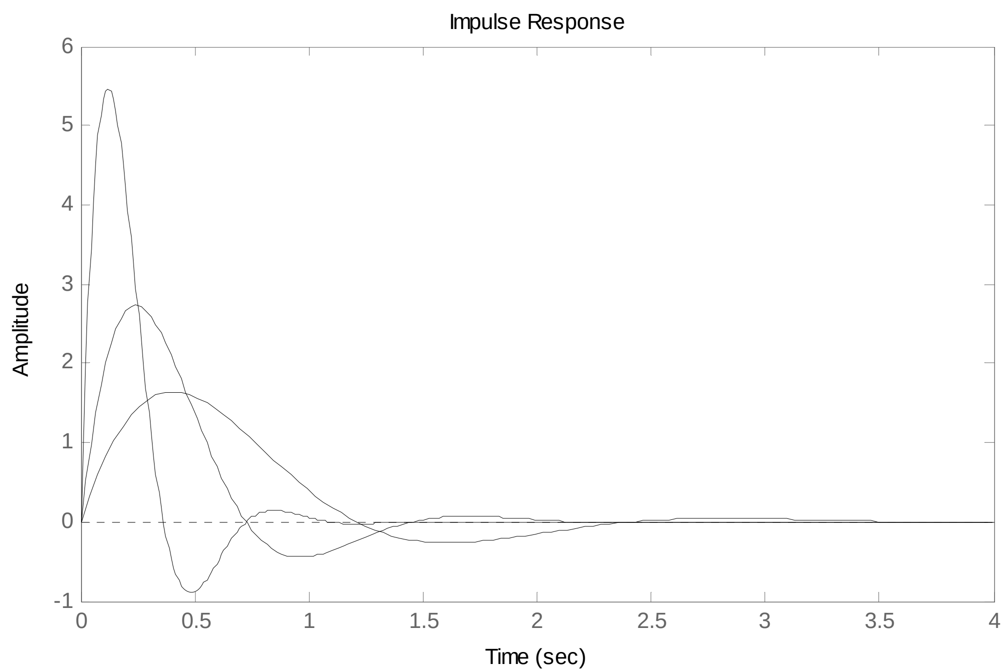
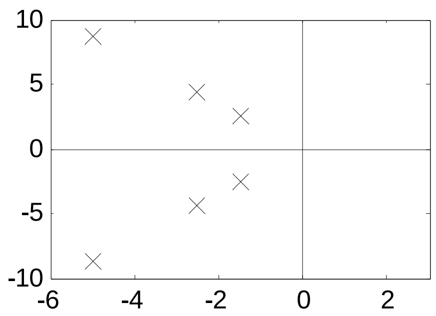


Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ

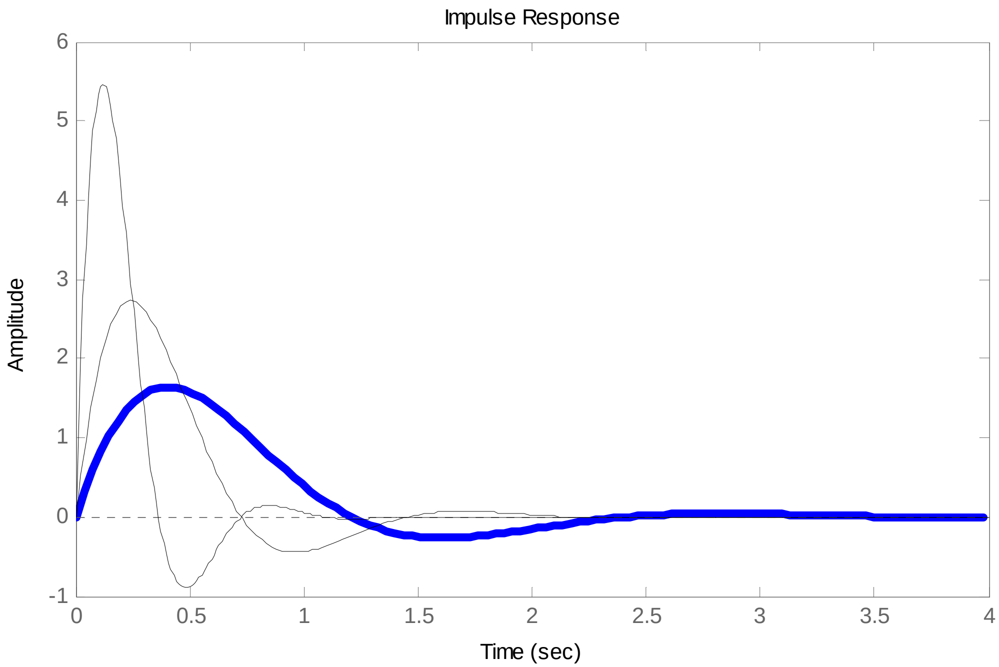


$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$

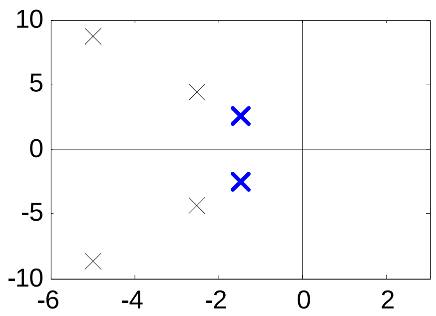


Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ

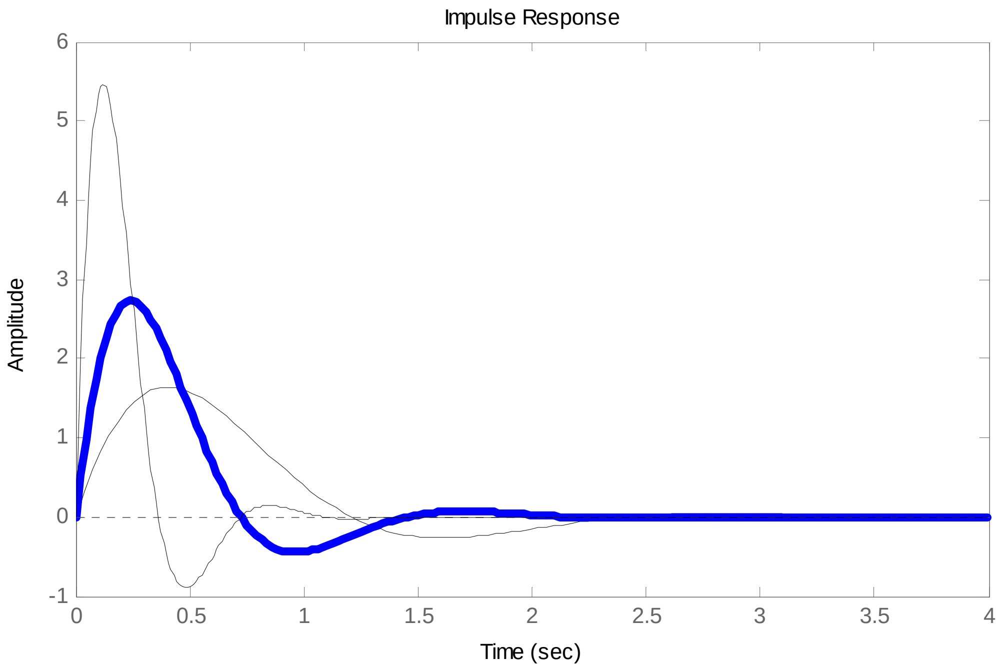


$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$



Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ



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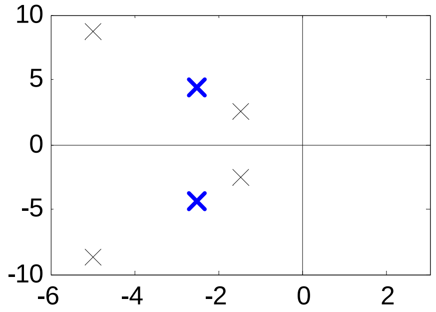
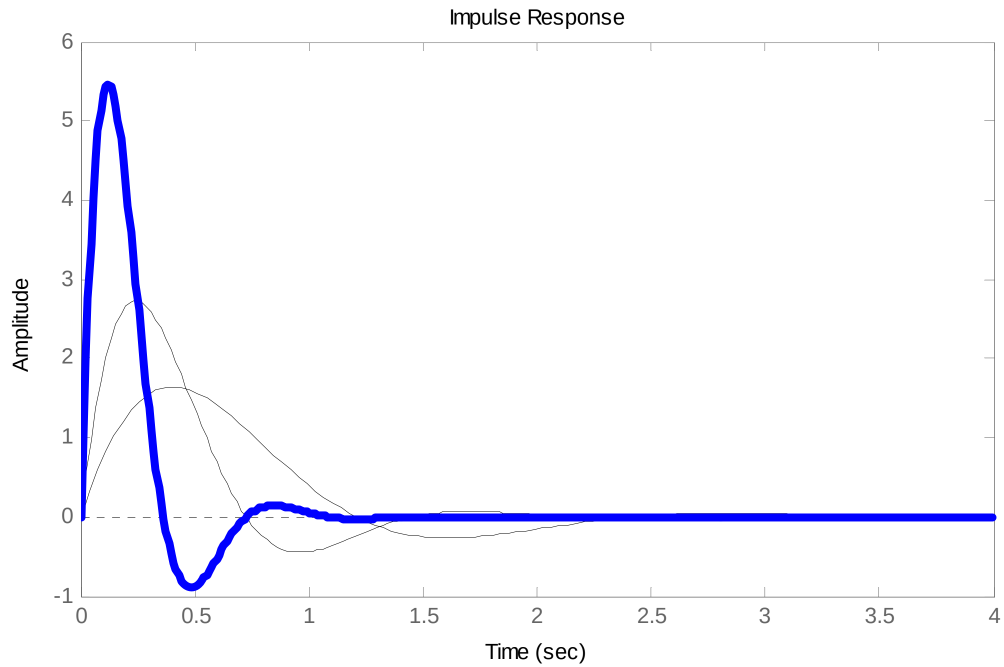


Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ



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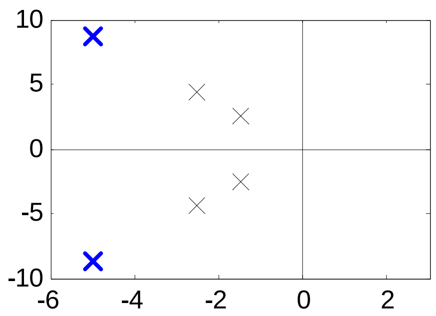
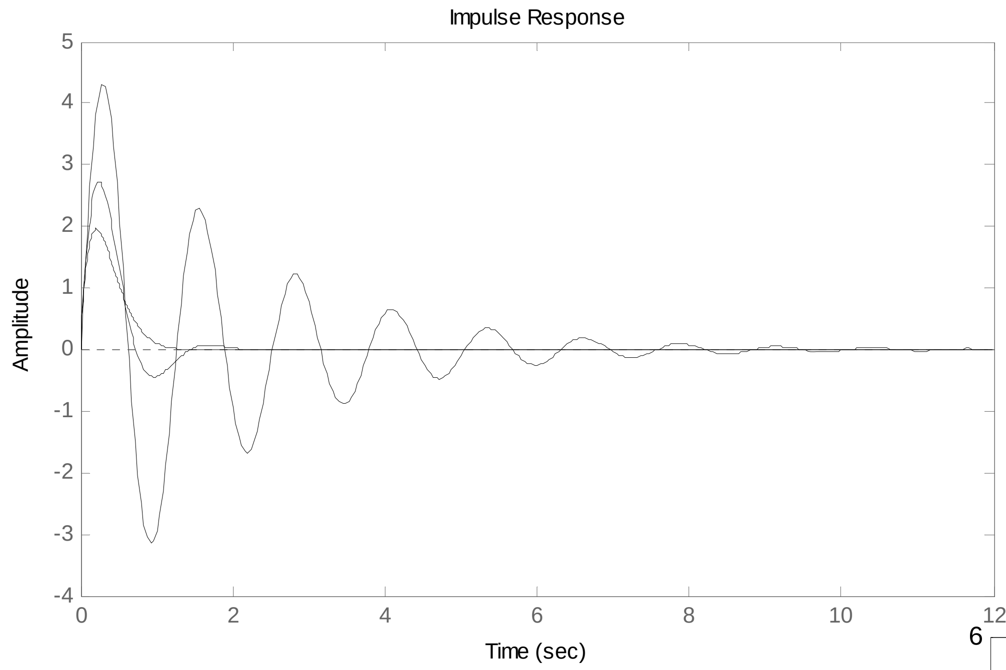
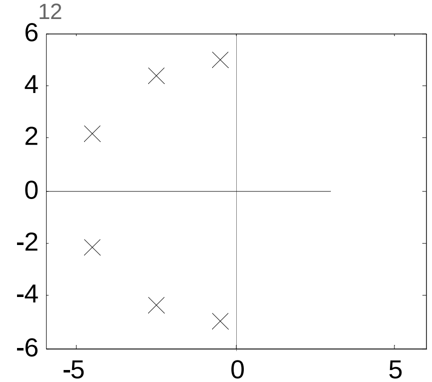


Second order impulse response – Underdamped and Undamped

Increasing ξ / Fixed ω_n

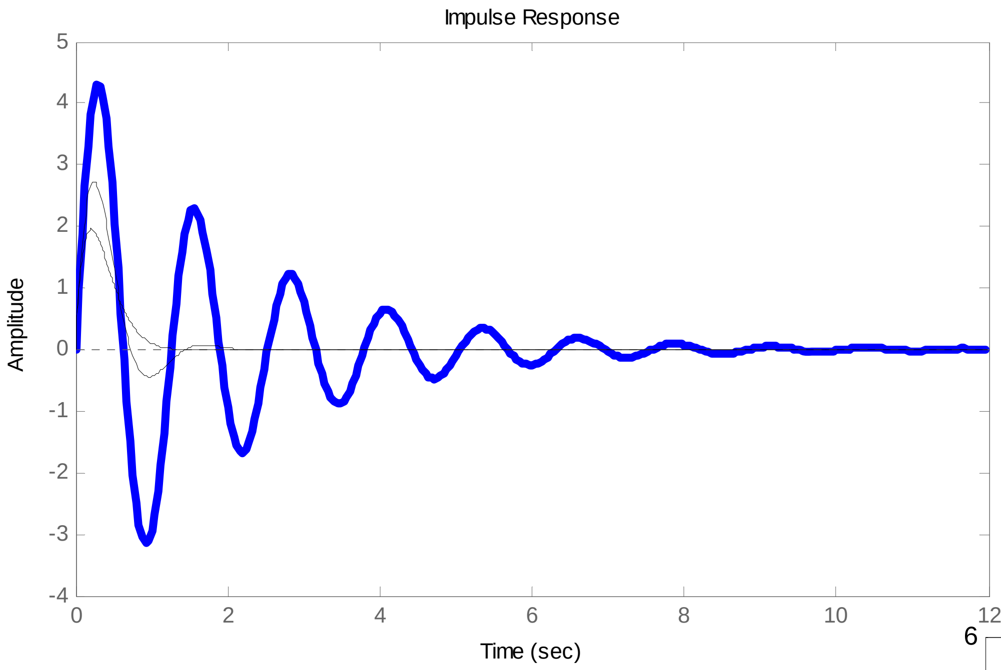




$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}t)1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing ξ / Fixed ω_n



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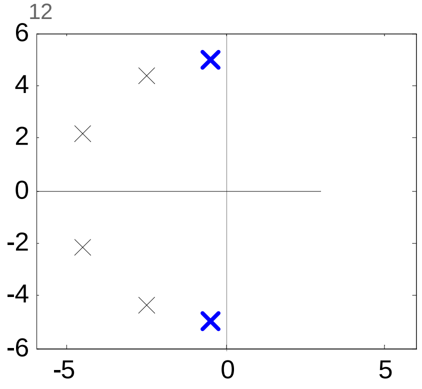
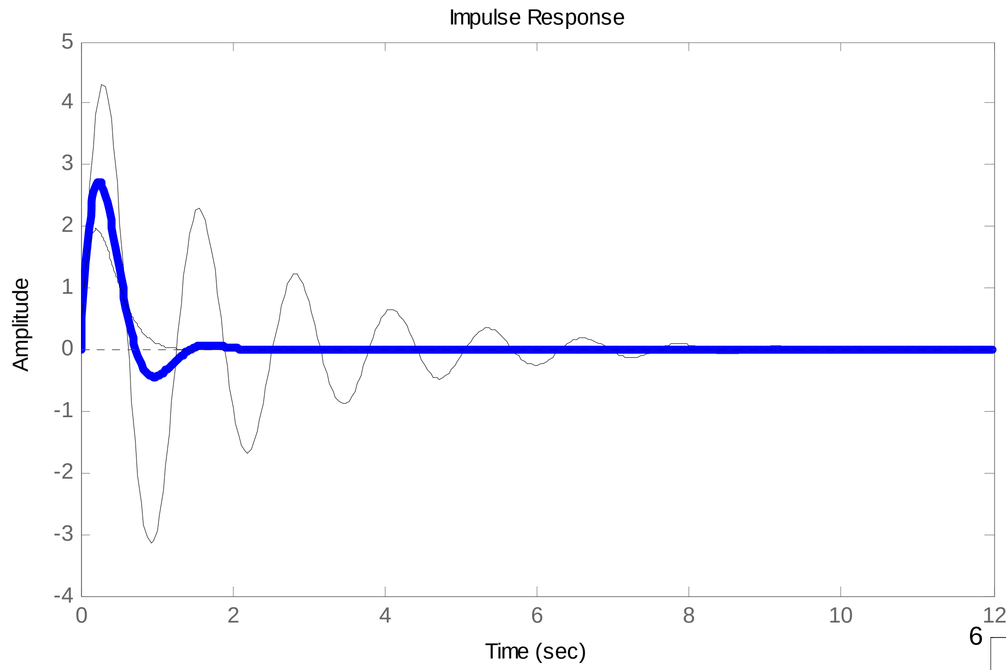
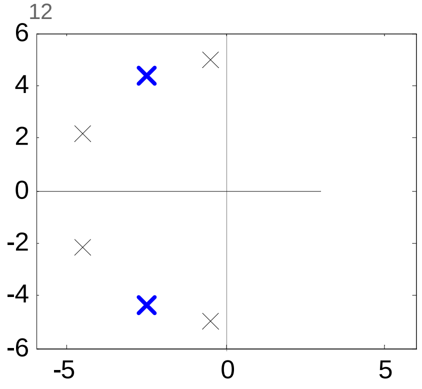


Second order impulse response – Underdamped and Undamped

Increasing ξ / Fixed ω_n

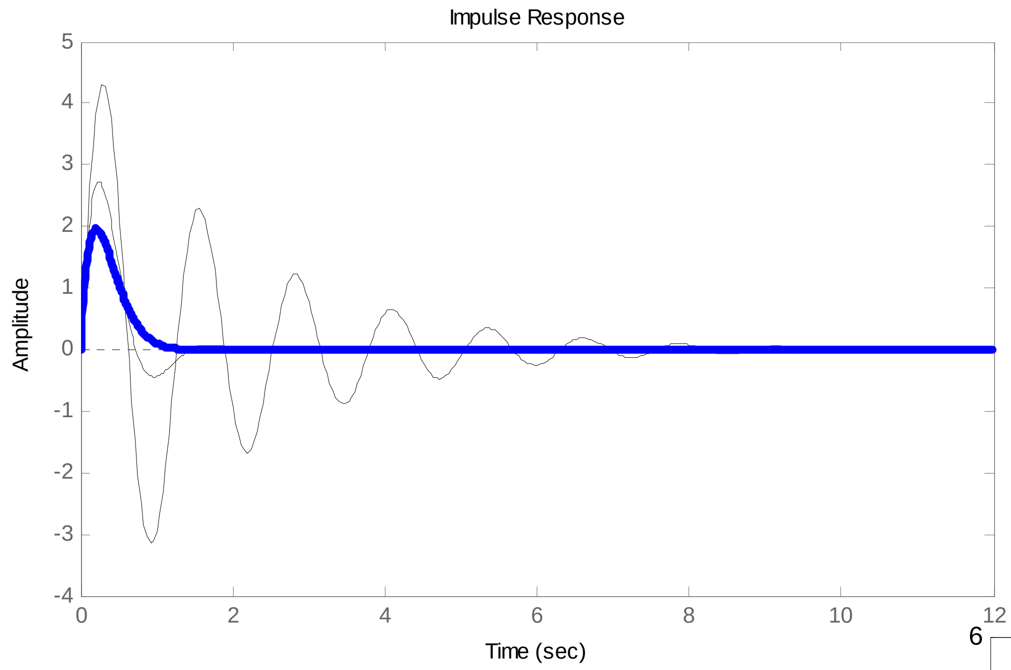




$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}t)1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing ξ / Fixed ω_n



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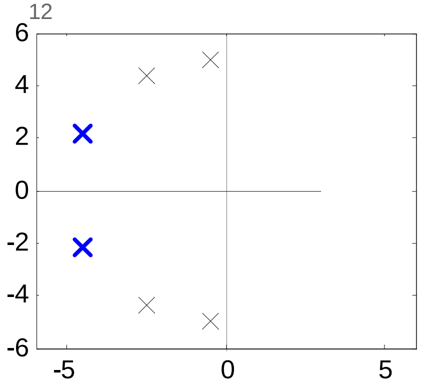
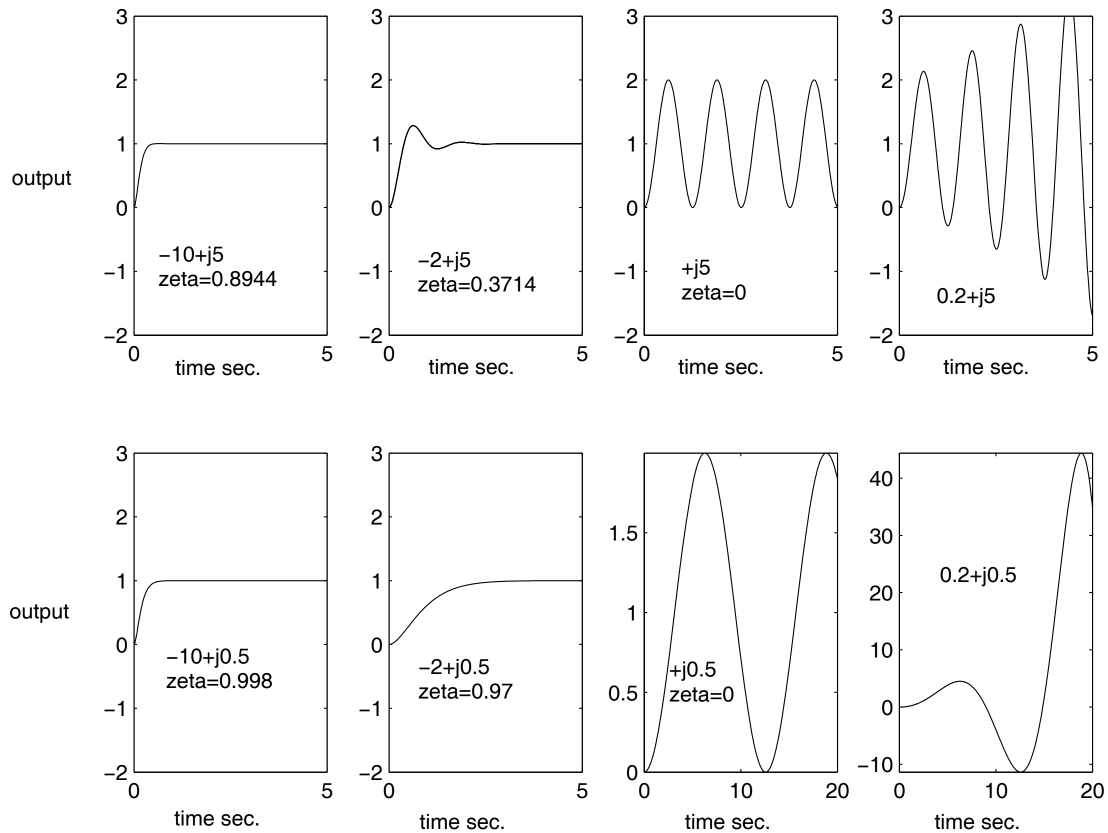
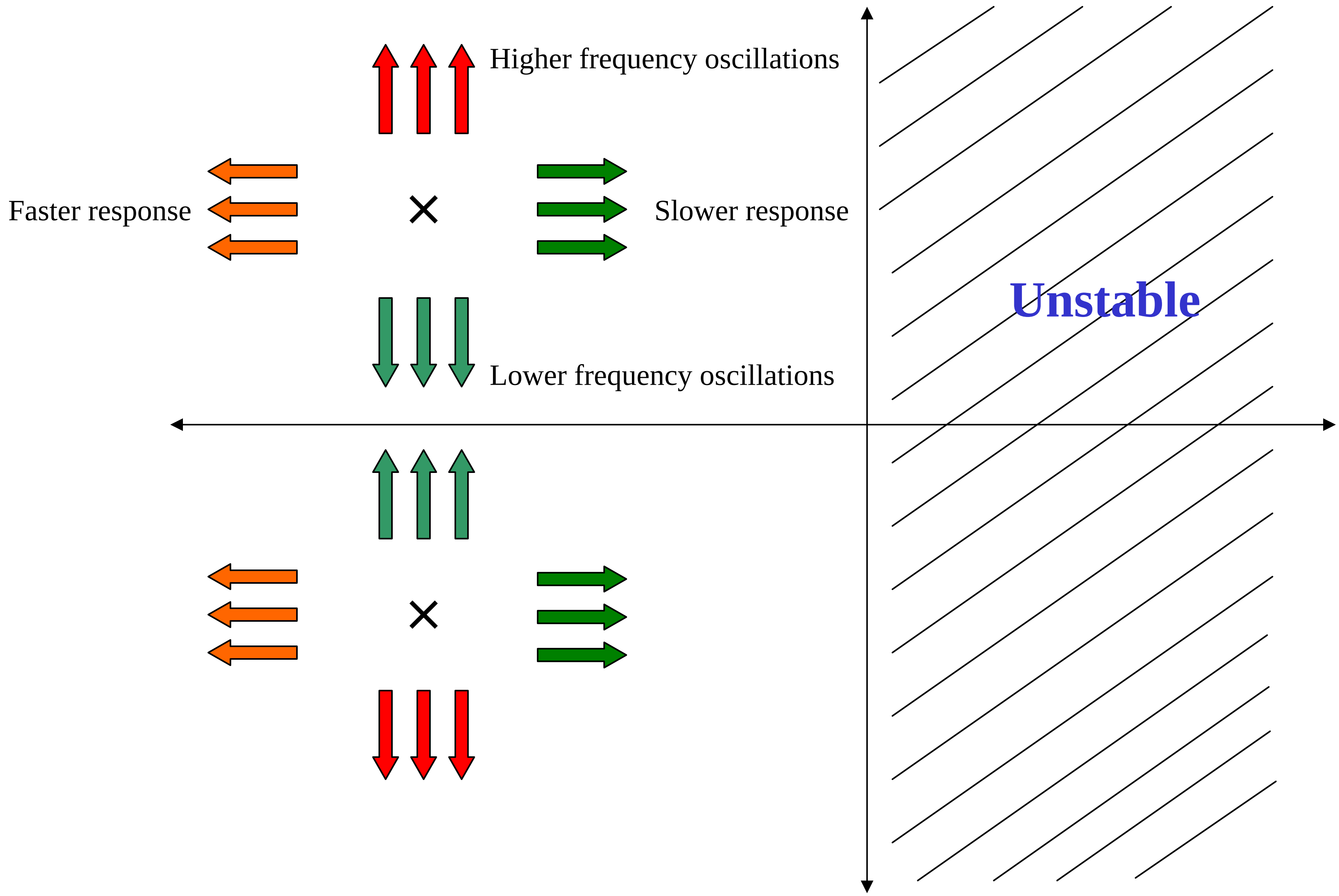


Second order step response – Underdamped and Undamped

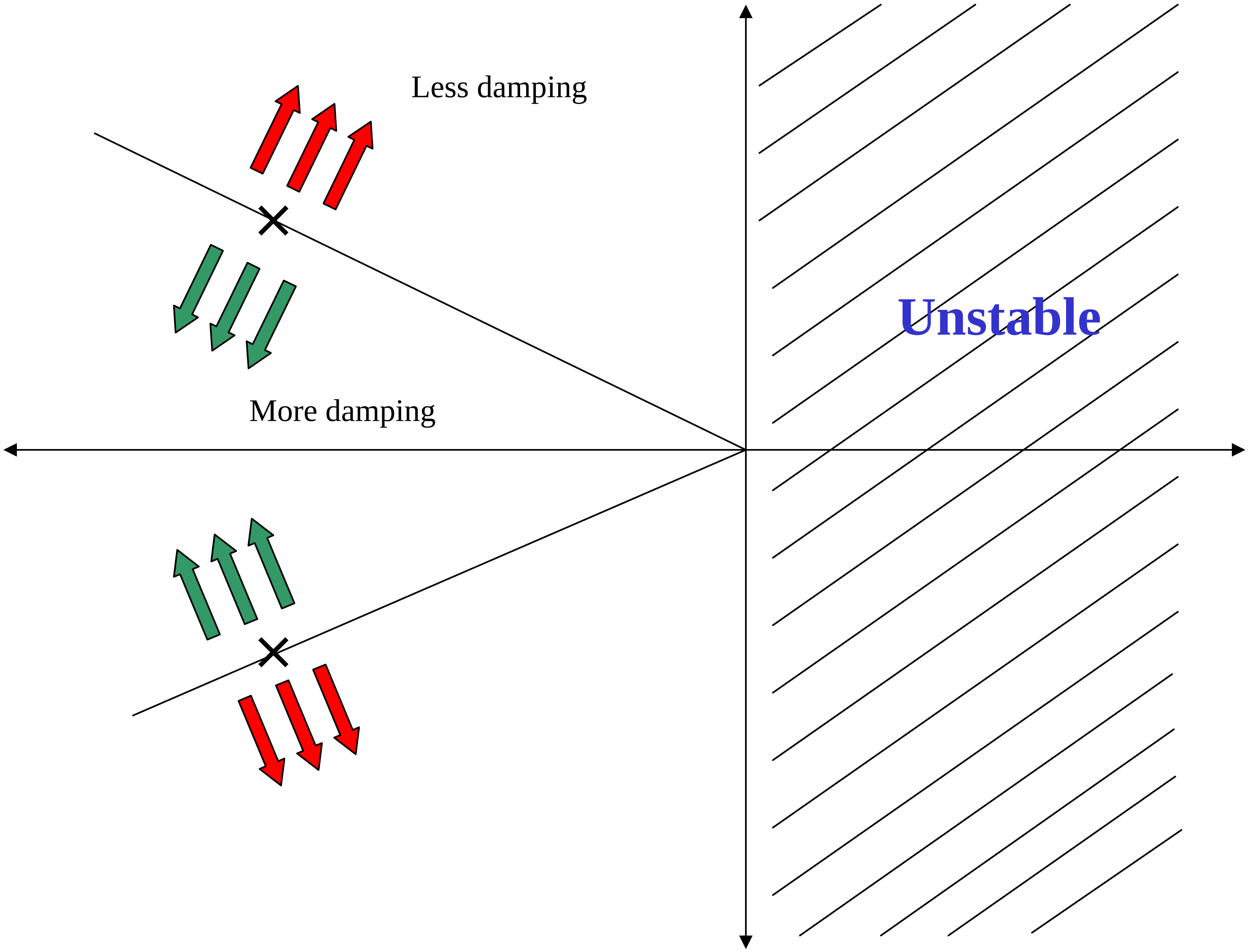


$$y_{step}(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\omega_n t + \theta)\right) 1(t)$$

Second order impulse response – Underdamped and Undamped

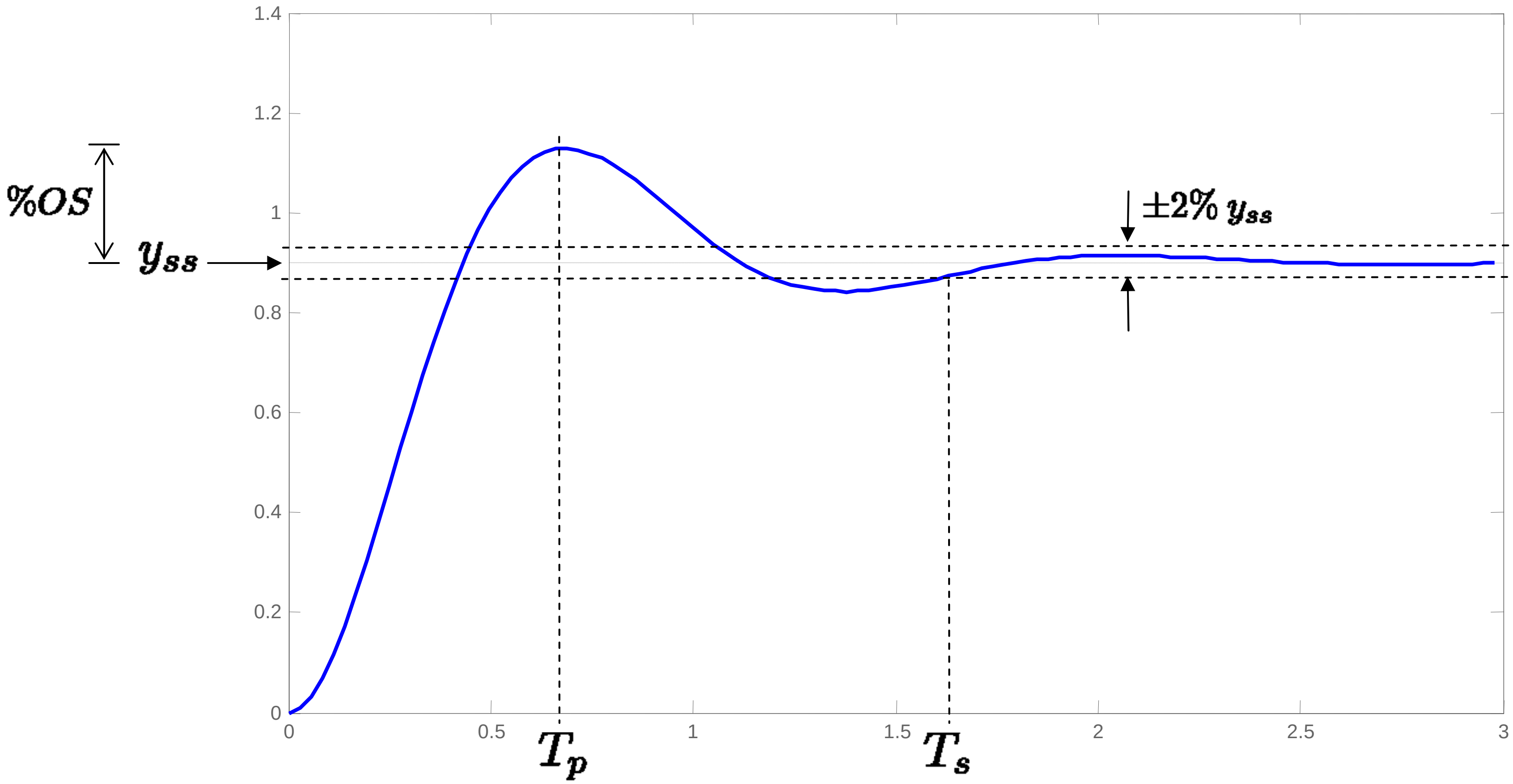


Second order impulse response – Underdamped and Undamped



Second order step response – Time specifications.

$$1(t) \longrightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\xi\omega_n\, s + \omega_n^2}} \longrightarrow y_{step}(t)$$



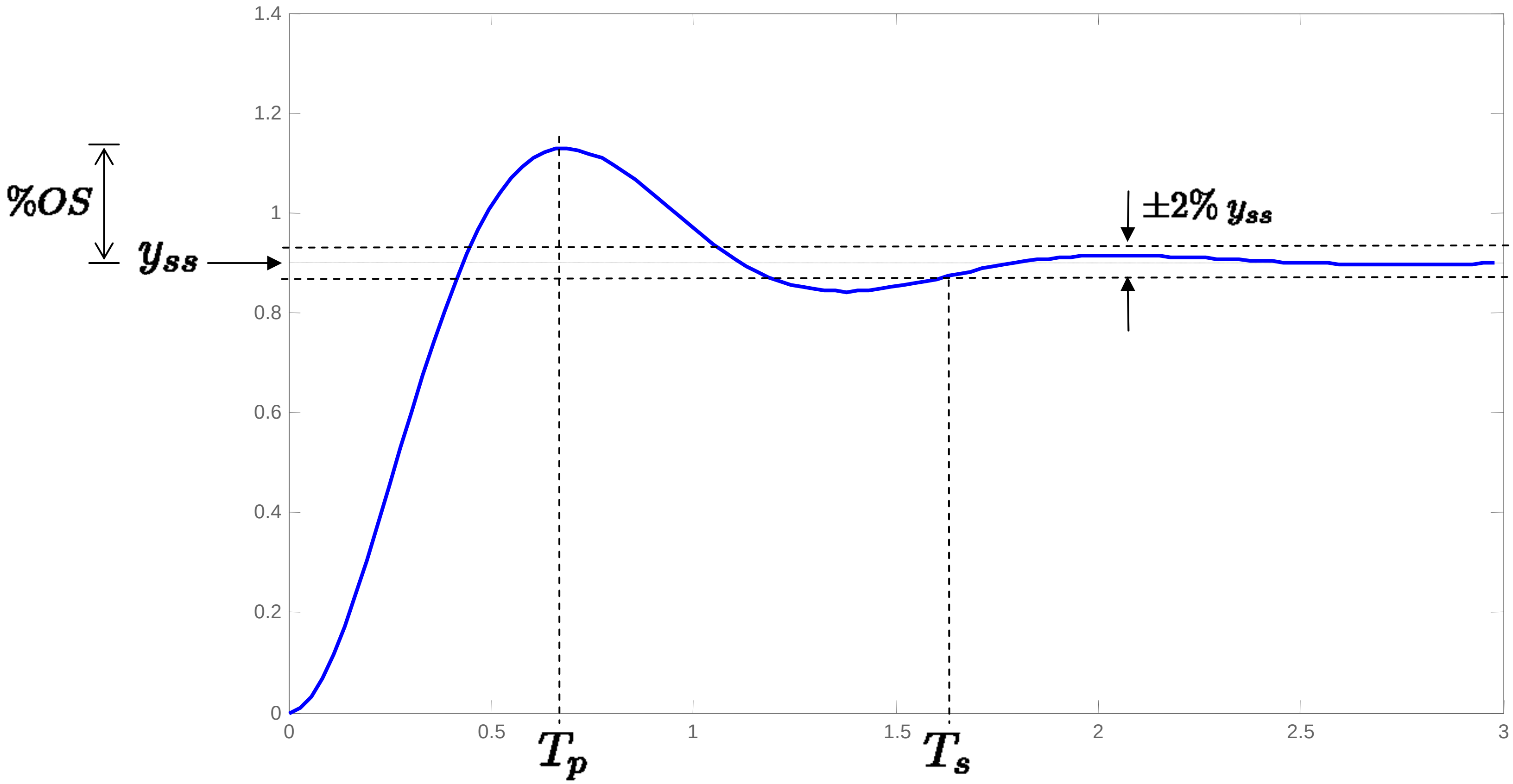
Second order step response – Time specifications.

y_{ss} … Steady state value.

T_p … Time to reach first peak (undamped or underdamped only).

$\%OS$ … % of $y_{step}(T_p)$ in excess of y_{ss} .

T_s … Time to reach and stay within 2% of y_{ss} .



Second order step response – Time specifications.

- y_{ss} … **Steady state value**.

$$y_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\left(\frac{1}{s}\right)H(s) = \lim_{s\to 0} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2} = 1$$

More generally, if the numerator is not ω_n^2, but some K :

$$H(s) = \frac{K}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \Rightarrow \quad \boxed{y_{ss} = \frac{K}{\omega_n^2}}$$

Second order step response – Time specifications.

- T_p … **Peak time**.

$$\dot{y}_{step} = \mathcal{L}^{-1}\Big[\, s\,Y(s) \Big] = \mathcal{L}^{-1}\Big[\, s\,\frac{1}{s}\,H(s) \Big] = \mathcal{L}^{-1}\Big[\, H(s) \Big]$$

$$= \mathcal{L}^{-1}\Big[\, \frac{K}{s^2 + 2\xi\omega_n s + \omega_n^2} \Big]$$

$$= \frac{K}{\omega_n\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t}\, \sin(\omega_n\sqrt{1-\xi^2}\, t)\, 1(t)$$

Therefore,

$$\dot{y}_{step} = 0 \quad \Leftrightarrow \quad \sin(\omega_n\sqrt{1-\xi^2}\, t) = 0$$

$$\Leftrightarrow \quad t = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}$$

T_p is the time of the occurrence of the first peak $(n = 1)$:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

Second order step response – Time specifications.

- $\%OS$ … **Percent overshoot**.

$$y_{step}(t) \;=\; \frac{K}{\omega_n^2}\left[\,1-e^{-\xi\omega_n t}\left(\cos(\omega\,t)+\frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega\,t)\right)\,\right]1(t)$$

Evaluating at T_p ,

$$y_{step}(T_p) \;=\; \frac{K}{\omega_n^2}\left(1+e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}\right)$$

$\%OS$ is defined as:

$$\%OS = 100 \times \frac{y_{step}(T_p)-y_{ss}}{y_{ss}}$$

Substituting our expressions for $y_{step}(T_p)$ and y_{ss} :

$$\boxed{\%OS = 100 \times e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}$$

Second order step response – Time specifications.

- T_s … **Settling time**.

Defining θ with $\xi = \sin(\theta)$, the previous expression for $y_{step}(t)$ can be re-written as:

$$y_{step}(t) = \frac{K}{\omega_n^2}\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cos(\omega\, t - \theta)\right] 1(t)$$

As an approximation, we find the time it takes for the exponential envelope to reach 2% of y_{ss} .

$$\frac{K}{\omega_n^2\sqrt{1-\xi^2}} e^{-\xi\omega_n T_s} = 0.02$$

$$\Rightarrow \qquad T_s = \frac{-\ln\left(0.02\sqrt{1-\xi^2}\,\omega_n^2/K\right)}{\xi\omega_n}$$

$$\Rightarrow \qquad \boxed{T_s \approx \frac{4}{\xi\omega_n} \quad \text{when} \quad K = \omega_n^2}$$

Typical specifications for second order systems.

How many independent parameters can we specify?

$$H(s) = \frac{K}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \Rightarrow \quad \mathbf{3}$$

$$y_{ss} = \frac{K}{\omega_n^2} \longrightarrow 1 - \epsilon \leq y_{ss} \leq 1 + \epsilon$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}} \longrightarrow T_p \leq T_p^{max}$$

$$\%OS = 100 \times e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \longrightarrow \%OS \leq \%OS^{max}$$

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\xi^2}\,\omega_n^2/K\right)}{\xi\omega_n} \longrightarrow T_s \leq T_s^{max}$$