MICHIGAN STATE UNIVERSITY
Department of Mechanical Engineering
ME451 Control Systems Spring 2007

Midterm Exam II

Closed Book. Two 8.5 × 11 pages of handwritten notes are allowed.

Your Name: Solution	
Student Number:	

Please answer all questions.

Problem:	1	2	Total
Max. Grade:	50	50	100
Grade:	50	50	100

1 Problem

a. (20 points) Determine the range of the controller gains (K, K_I) so that the PI feedback system in Fig. 1 is BIBO (or asymptotically) stable. Plot this range in a two-dimensional space. (Hint: Use the Routh-Hurwitz stability test.)

$e_{ss} = \frac{3}{4}$

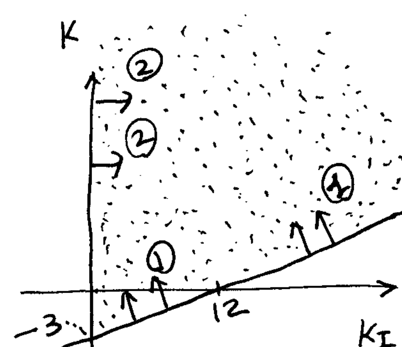
b. (10 points) Set gains at $K = 1, K_I = 0$. Consider a unit step input $R(s) = \frac{1}{s}$. Determine the steady state error $e_{ss} := \lim_{t\to\infty} e(t)$, where $e(t) := r(t) - y(t)$.

$e_{ss} = 0$

c. (10 points) Set gains at $K = 1, K_I = 0.01$. Consider a unit step input $R(s) = \frac{1}{s}$. Determine the steady state error $e_{ss} := \lim_{t\to\infty} e(t)$, where $e(t) := r(t) - y(t)$.

d. (10 points) Set gains at $K = 1, K_I = 0$. Consider a sinusoidal input signal $r(t) = 10\sin(2t)$. Determine the steady state signal $y_{ss}(t)$ for this input.

$y_{ss}(t) = \frac{10}{8}\sin\left(2t - \frac{\pi}{2}\right)$



Check the location of closed-loop poles

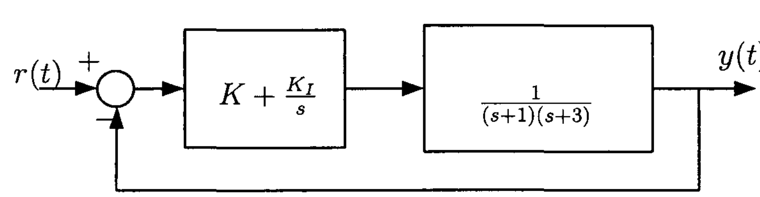


Figure 1: A system with a PI controller.

(a) C.E. ⇒ $1 + \left(K + \frac{K_I}{s}\right)\frac{1}{(s+1)(s+3)} = 0$

or $1 + \frac{sK + K_I}{s}\,\frac{1}{(s+1)(s+3)} = 0$

$\rightarrow s(s^2 + 4s + 3) + sK + K_I = 0$

C.E. $s^3 + 4s^2 + (3+K)s + K_I = 0$

Routh-Array

$$\begin{array}{lll} s^3 & 1 & 3+K \\ s^2 & 4 & K_I \\ s^1 & b & \\ s^0 & K_I > 0 & \text{— ②} \end{array}$$

$$b = -\frac{1}{4}\begin{vmatrix} 1 & 3+K \\ 4 & K_I \end{vmatrix} = -\frac{1}{4}(K_I - 12 - 4K) > 0 \quad \text{— ①}$$

$\rightarrow$ or $K_I - 12 - 4K < 0$

Stability condition

TF from r to e

ⓑ. $T_{r \to e} = \dfrac{1}{1 + \left(K + \frac{K_I}{s}\right) \frac{1}{(s+1)(s+3)}}$

$E(s) = R(s) \cdot T_{r \to e}(s)$

$(K=1,\ K_I = 0) \Rightarrow$ system is stable by ①, ②

use Final value theorem

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s \frac{1}{s}\left[\frac{1}{1 + (1) \cdot \frac{1}{(s+1)(s+3)}}\right] = \frac{1}{1+\frac{1}{3}} = \boxed{\frac{3}{4}}$$

ⓒ. $(K=1,\ K_I = 0.01)$ system is stable by ①, ② use F.V.T.

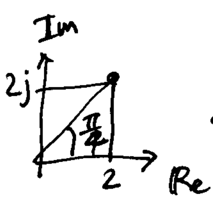
$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s \cdot \frac{1}{s}\left[\frac{1}{1 + \left(1 + \frac{0.01}{s}\right)\frac{1}{(s+1)(s+3)}}\right] = \frac{1}{\infty} = \boxed{0}$$

ⓓ.

T.F. from r to y

$$T_{r \to y}(s) = \frac{F_g}{1 - L_g} = \frac{\frac{1}{(s+1)(s+3)}}{1 + \frac{1}{(s+1)(s+3)}} = \frac{1}{s^2 + 4s + 4} = \frac{1}{(s+2)^2}$$

$$T_{ry}(j2) = \frac{1}{(2j+2)^2} = \frac{1}{\left(2\sqrt{2}\, e^{\frac{\pi}{4} j}\right)^2} = \underbrace{\frac{1}{4 \cdot 2}}_{\text{magnitude}}\ \underbrace{e^{-\frac{\pi}{2} j}}_{\text{phase}}$$



$2j + 2 = 2\sqrt{2}\, e^{\frac{\pi}{4} j}$

$$y_{ss}(t) = 10 \cdot \left(\frac{1}{4 \cdot 2}\right) \cdot \sin\left(2t - \frac{\pi}{2}\right)$$

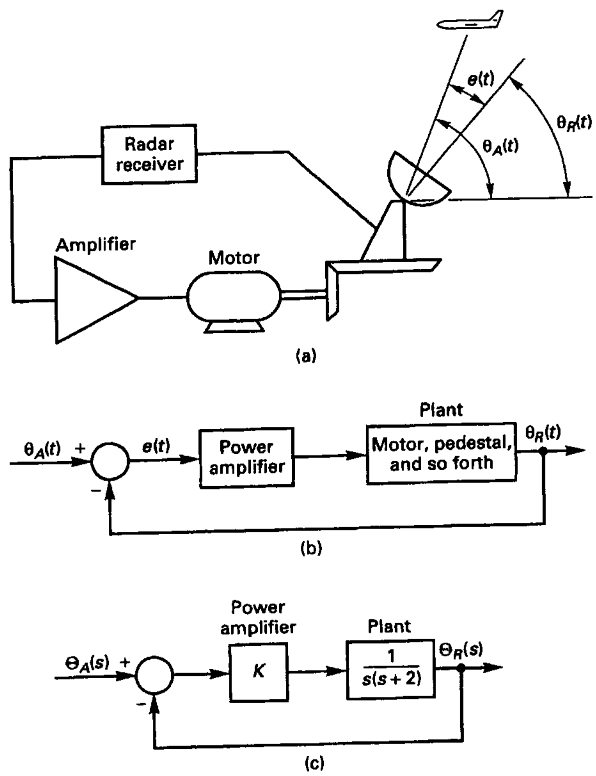
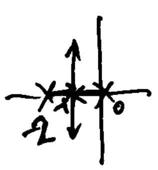


Figure 2: Radar Tracking System.

2 Problem

Consider the radar tracking system illustrated in Fig. 2. We need to design the tracking system so that the radar tracks the target airplane properly, i.e., $\theta_R(t)$ should be able to track $\theta_A(t)$ properly.



a. **(15 points)** Draw the root locus of the system. Specify the location of a break away point. (Hint: Obtain the closed loop poles in terms of K. Plot them as you vary K from 0 to ∞ in s-plane.)

$1 \leq K \leq 2$

b. **(20 points)** An airforce Colonel demands the following time specifications of the closed-loop second order system with a step input $\theta_A(t)$:

- Spec. 1 the system has to be either critically damped or underdamped for fast dynamics,
- Spec. 2 the maximum allowable settling time T_s is 4 sec. (Hint: use $T_s = \frac{4}{\zeta\omega_n}$.),
- Spec. 3 the maximum allowable percentage overshoot is 4.32 %. (Hint: this is achieved by $\zeta = 0.707$.)

Determine the range of K for which the closed-loop system satisfies the time specifications.

$e_{ss} = 1$

c. **(15 points)** Fix $K = 2$. For a unit ramp input $\Theta_A(s) = \frac{1}{s^2}$, determine the steady state error $e_{ss} := \lim_{t\to\infty} e(t)$, where $e(t) = \theta_A(t) - \theta_R(t)$ as shown in Fig. 2-(b).

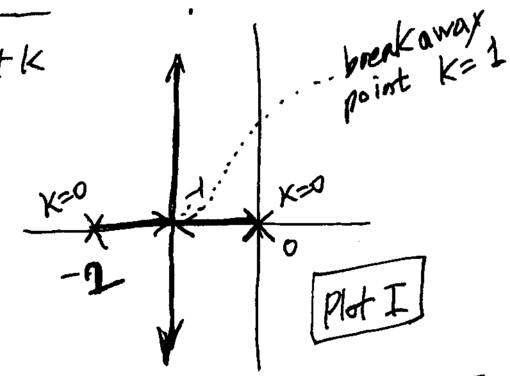
T.F. From θ_A to θ_R

$$T(s) = \frac{K\frac{1}{s(s+2)}}{1+K\frac{1}{s(s+2)}} = \frac{K}{s^2+2s+K}$$

(a)

C.E. $\quad s^2+2s+K=0$

$$s_{1,2} = -1 \pm \sqrt{1-K}$$



$$\begin{cases} 0 \le K < 1 & \rightarrow \text{ over damped system } \cancel{\text{with spec. 1}} \quad (s_{1,2} = -1 \pm \sqrt{1-K}) \\ K = 1 & \rightarrow \text{ critically damped system } \quad (\text{two roots at } -1) \\ 1 < K < \infty & \rightarrow \text{ under damped system } \quad (s_{1,2} = -1 \pm \sqrt{K-1}\, j) \end{cases}$$

(b). From Spec 1. $\rightarrow \quad K \ge 1$

From Spec 2. $\rightarrow \quad T_s = \frac{4}{\zeta\omega_n} \le 4 \rightarrow \boxed{-\zeta\omega_n \le -1}$

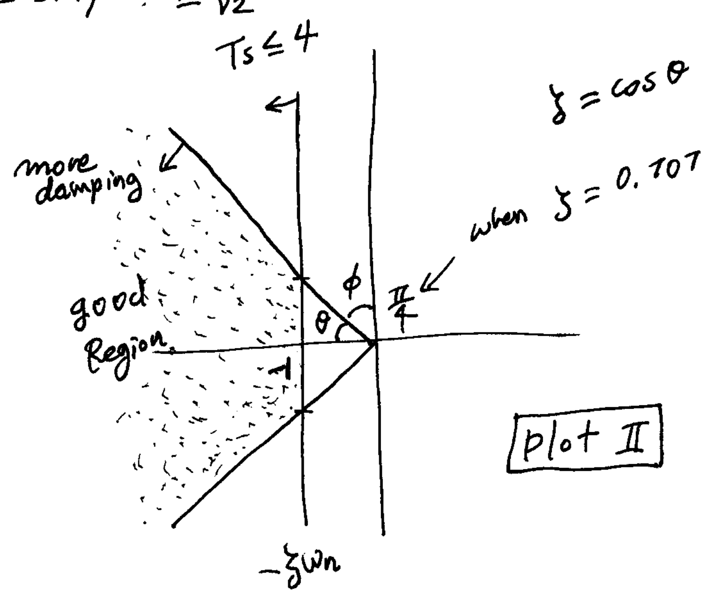
From Spec 3. $\rightarrow \quad \zeta = \cos\theta = \sin\phi \ge \frac{1}{\sqrt{2}}$

Range of K satisfies Spec 1.–3.

$$\boxed{1 \le K \le 2}$$

when $K = 2$

$$s_{1,2} = -1 \pm j$$



(c)

$$T_{\theta_A \to e} = \frac{1}{1+K\frac{1}{s(s+2)}}$$

$K = 2 \rightarrow$ stable

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s \cdot \frac{1}{s^2}\left[\frac{1}{1+\frac{2}{s(s+2)}}\right] = \lim_{s\to 0} \frac{1}{s+\frac{2}{(s+2)}} = \boxed{1}$$