Transfer Function: Laplace Transform of Unit Impulse Response of the System

- Input signal: $u(t) = e^{st}$ where $t \in (-\infty, \infty)$
- Output signal: $y(t) = u(t) * h(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau$

$$= \int_0^\infty h(\tau)e^{s(t-\tau)}d\tau$$

$$= \int_0^\infty h(\tau)e^{st}e^{-s\tau}d\tau$$

$$= \underbrace{\left[\int_0^\infty h(\tau)e^{-s\tau}d\tau\right]}_{=:\mathcal{L}(h)=:H(s)} e^{st}$$

$$=: H(s)e^{st}$$

def: Transfer Function

Laplace transform of the impulse response

- Take $s = jw$

$$u(t) = e^{jwt} = \cos(wt) + j\sin(wt) \rightarrow \boxed{H(jw)} \rightarrow y(t) = H(jw)e^{jwt}$$

Frequency Response

- Input $A\cos wt = \frac{A}{2}(e^{jwt} + e^{-jwt})$
- We know

$$e^{jwt} \rightarrow \boxed{H(s)} \rightarrow H(jw)e^{jwt}$$

$$e^{-jwt} \rightarrow \boxed{H(s)} \rightarrow H(-jw)e^{-jwt}$$

- Complex numbers $H(jw) = Me^{j\phi}, \; H(-jw) = Me^{-j\phi}$

$$y(t) = \frac{A}{2}(H(jw)e^{jwt} + H(-jwt)e^{-jwt})$$

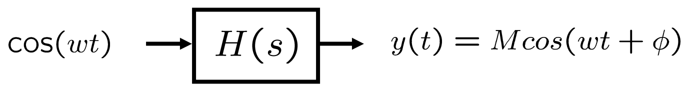
$$= \frac{A}{2}M(e^{j(wt+\phi)} + e^{-j(wt+\phi)})$$

$$= AMcos(wt + \phi)$$

Magnitude

Phase shift

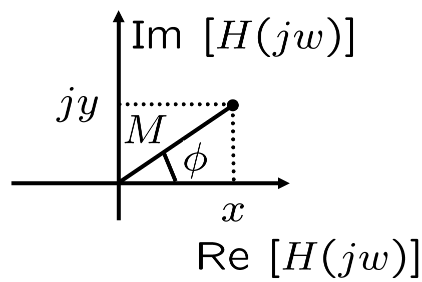
Frequency Response



$$H(jw) = Me^{j\phi} = x + jy$$

$$M = |H(jw)|$$

$$\phi = \tan^{-1}\left[\frac{\text{Im}[H(jw)]}{\text{Re}[H(jw)]}\right]$$



The Laplace Transform (Appendix B)

- Laplace transform converts a calculus problem (the linear differential equation) to an algebra problem
- How to Use it:
  - Take the Laplace transform of a linear differential equation
  - Solve the algebra problem
  - Take the Inverse Laplace transform to obtain the solution to the original differential equation

def: Laplace transform $F(s) := \mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt$

def: Inverse Laplace transform

$$f(t) := \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

The Laplace Transform (Appendix B)

- Laplace Transform of a function f(t)

$$F(s) := \mathcal{L}f(t) = \int_0^\infty f(t)e^{-st}dt$$

- Convolution integral

$$\begin{aligned} f_1(t) * f_2(t) &= \int_0^t f_1(\tau)f_2(t-\tau)d\tau \\ &= \int_0^t f_2(\tau)f_1(t-\tau)d\tau \end{aligned}$$

$$\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

Properties of Laplace Transforms (page 641-643)

- Linearity

$$\begin{aligned} \mathcal{L}[af(t) + bg(t)] &= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] \\ &= aF(s) + bG(s) \end{aligned}$$

- Time Delay

$$\begin{aligned} \mathcal{L}[f(t-\lambda)] &= \int_0^\infty f(t-\lambda)e^{-st}dt \\ &= \int_0^\infty f(\tau)e^{-s(\tau+\lambda)}d\tau \\ &= e^{-s\lambda}\mathcal{L}[f(t)] \end{aligned}$$

Non-rational function

Properties of Laplace Transforms

- Shift in Frequency

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-at}f(t)e^{-st}dt$$
$$= F(s+a)$$

- Differentiation

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}dt$$
$$= e^{-st}f(t)|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$
$$= sF(s) - f(0)$$

Properties of Laplace Transforms

- Differentiation (d/dt in time domain ⇔ s in Laplace domain)

$$\mathcal{L}[\dot{f}] = s\mathcal{L}[f] - f(0)$$

$$\mathcal{L}[\ddot{f}] = s\mathcal{L}[\dot{f}] - \dot{f}(0)$$
$$= s^2\mathcal{L}[f] - sf(0) - \dot{f}(0)$$

- Integration ($\int dt$ in time domain ⇔ 1/s in Laplace domain)

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

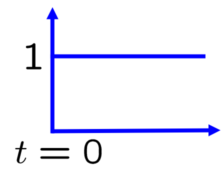
Laplace Transform of Impulse and Unit Step

- Impulse

$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = 1$$

- Unit Step

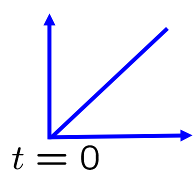
$$u(t) = 1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}[1(t)] = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

Unit Ramp

$$f(t) = t, \forall t \geq 0$$



$$\mathcal{L}(t) = \int_0^\infty t e^{-st} dt = \frac{1}{s^2}$$

Exponential Function

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt = \frac{-e^{-(s-a)t}}{s-a}\Big|_0^\infty$$
$$= \frac{1}{s-a}, \quad \text{Re}[s-a] > 0$$

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}}$$

Sinusoidal Functions $f(t) = \sin(wt)$, or $\cos(wt)$

$$\boxed{e^{jwt} = \cos(wt) + i\sin(wt)}$$

$$\mathcal{L}\left[e^{jwt}\right] = \frac{1}{s-jw} = \frac{s+jw}{s^2+w^2}$$
$$= \mathcal{L}[\cos(wt)] + j\mathcal{L}[\sin(wt)]$$

$$\boxed{\mathcal{L}[\cos(wt)] = \frac{s}{s^2+w^2}}$$

$$\boxed{\mathcal{L}[\sin(wt)] = \frac{w}{s^2+w^2}}$$

Partial-fraction Expansion (Text, page 637-641)

- F(s) is rational, $m \leq n \longrightarrow$ realizable condition (d/dt is not realizable)

$$
\begin{aligned}
F(s) &= \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\
&= \frac{b_m \prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} \\
&= \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}
\end{aligned}
$$

zeros: $(s - z_i)$

poles: $(s - p_i)$

$$f(t) = \sum_{i=1}^{n} k_i e^{p_i t} 1(t)$$