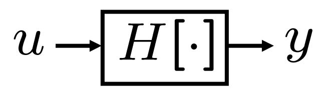
Output Signal of a Linear System

- Input signal $u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau,$
- Output signal



$$
\begin{aligned}
y(t) &= H[u(t)] \\
&= H\left[\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} u(\tau)H[\delta(t-\tau)]d\tau \\
&=: \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \\
&=: u(t) * h(t) \\
&= \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau \\
&= \int_{0}^{\infty} u(t-\tau)h(\tau)d\tau
\end{aligned}
$$

Superposition!

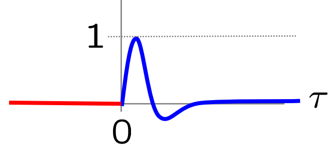
def: impulse response

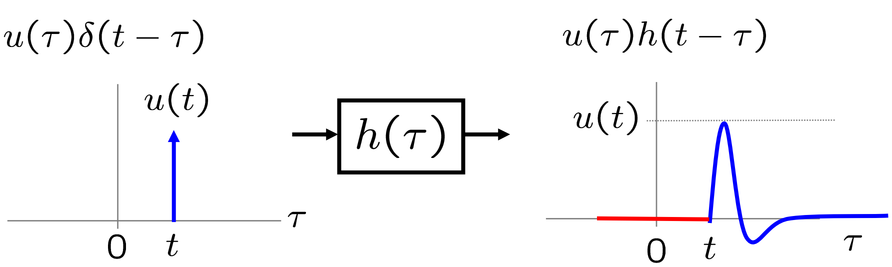
def: convolution

def: causality

Impulse Response

$$h(\tau) := H(\delta(\tau))$$

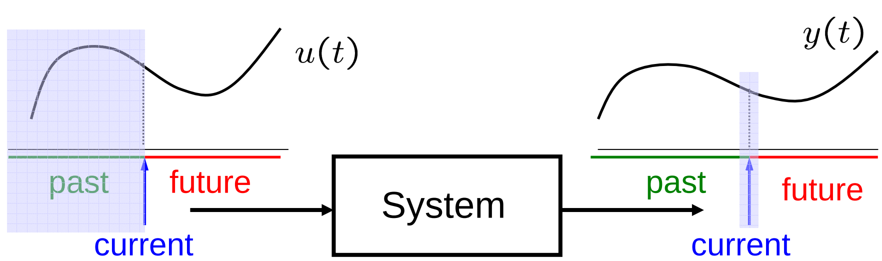




Causal Linear Time Invariant (LTI) System

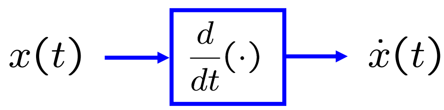
- A **causal system** (a physical or nonanticipative system) is a system where the output $y(t)$ only depends on the input values $\{u(\tau) | -\infty < \tau \leq t\}$
- Thus, the current output $y(t)$ can be generated by the causal system with the current and past input values
- Causal LTI impulse response
  $h(\tau) = 0, \text{ for } \tau < 0$
- Thus, we have
  $$y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_{0}^{\infty} u(t-\tau)h(\tau)d\tau$$

Causal System (Physically Realizable)

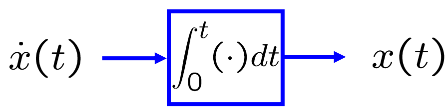


Causal System?

- Derivative operator (input: position, output: velocity)



- Integral operator (input: velocity, output: position)



Complex Numbers

- Ordered pair of two real numbers

$$s := x + jy \in \mathcal{C}, \text{where } x, y \in \mathcal{R}, j = \sqrt{-1}$$

- Conjugate $\bar{s} = s^* := x - jy$
- Addition $s_1 = x_1 + jy_1, s_2 = x_2 + jy_2$

$$s_1 + s_2 = (x_1 + x_2) + j(y_1 + y_2)$$

- Multiplication

$$s_1 s_2 = (x_1 + jy_1)(x_2 + jy_2)$$
$$= (x_1 x_2 - y_1 y_2) + j(y_1 x_2 + x_1 y_2)$$

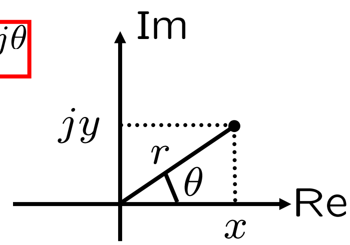
$$\boxed{ss^* = |s|^2 = x^2 + y^2}$$

Complex Numbers

- Euler's identity $$e^{j\theta} := \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}, \sin\theta = \frac{e^{j\theta}+e^{-j\theta}}{2j}$$

- Polar form $$s := x + jy = re^{j\theta}$$
- Magnitude $$r = \sqrt{x^2+y^2}$$
- Phase $$\theta = \tan^{-1}(y/x)$$



$$s_1 = r_1 e^{j\theta_1}, \; s_2 = r_2 e^{j\theta_2}$$

$$s_1 s_2 = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$\frac{s_1}{s_2} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

e^x, $\sin x$, $\cos x$

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$

$$\sin x := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Transfer Function: Laplace Transform of Unit Impulse Response of the System

- Input signal: $u(t) = e^{st}$ where $t \in (-\infty, \infty)$
- Output signal: $y(t) = u(t) * h(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau$

$$= \int_0^\infty h(\tau)e^{s(t-\tau)}d\tau$$

$$= \int_0^\infty h(\tau)e^{st}e^{-s\tau}d\tau$$

$$= \underbrace{\left[\int_0^\infty h(\tau)e^{-s\tau}d\tau\right]}_{=:\mathcal{L}(h)=:H(s)} e^{st}$$

$$=: H(s)e^{st}$$

def: Transfer Function

Laplace transform of the impulse response

- Take $s = jw$

$u(t) = e^{jwt} = \cos(wt) + j\sin(wt)$ → $H(jw)$ → $y(t) = H(jw)e^{jwt}$