New office Hours : [W: 10–11 am
5–6:30 PM]

Read Transformers & Gears Chapter 2.11
page 53–55

Linearization

state

① — $\dot{x} = f(x, u)$ nonlinear syst

input

Let u_o be a nominal input and let the resultant state be x_o

Let $u(\cdot) = u_o(\cdot) + \delta u(\cdot)$ be a perturbation
& the resultant perturbation
$x(\cdot) = x_o(\cdot) + \delta x(\cdot)$

Taylor series expansion

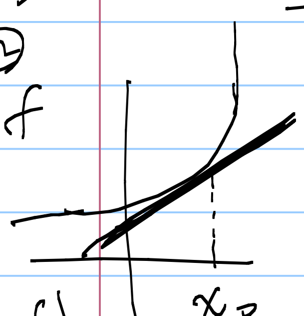
$$\begin{cases} f = ax + bu \\ \frac{\partial f}{\partial x} = a \end{cases} \qquad \frac{\partial f}{\partial u} = b$$

$$f(x, u) = f(x_o + \delta x, u_o + \delta u)$$

$$= f(x_o, u_o) + \frac{\partial}{\partial x} f(x, u)\Big|_{x = x_o, u = u_o} \cdot \delta x$$

$$+ \frac{\partial f(x, u)}{\partial u}\Big|_{x = x_o, u = u_o} \cdot \delta u + \underbrace{HOT}_{\approx 0} \quad ②$$

②



$\frac{\partial f}{\partial x}\Big|_{x = x_o}$

$\dot{x} = f(x, u) \rightarrow$

by ①, ② ③

$$\left.\begin{aligned} x &= x_0 + \delta x \\ u &= u_0 + \delta u \end{aligned}\right\} \text{— ③}$$

$\hookrightarrow \dot{x} = f(x, u)$

$$\cancel{\dot{x}_0} + \delta \dot{x} = \cancel{f(x_0, u_0)} + \frac{\partial}{\partial x} f(x,u)\Big|_{x_0 u_0} \cdot \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{x_0 u_0} \delta u$$

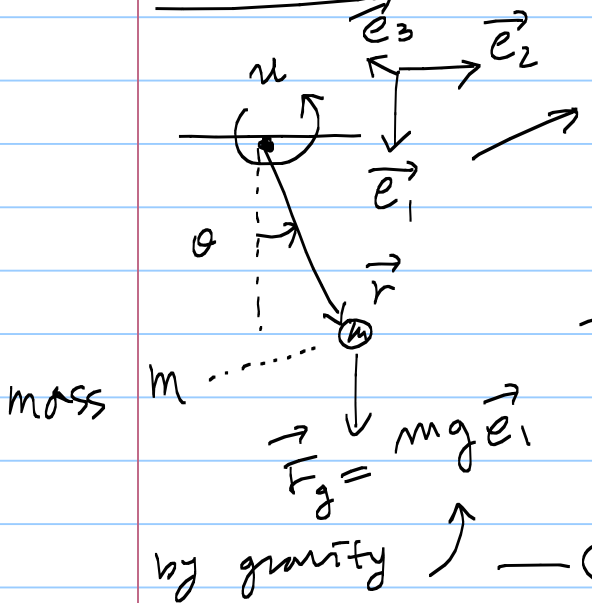
Should also satisfy

$\dot{x}_0 = f(x_0, u_0)$

$$\boxed{\delta \dot{x} = \frac{\partial f}{\partial x}\Big|_{x_0 u_0} \cdot \delta x + \frac{\partial f}{\partial u}\Big|_{x_0 u_0} \delta u} \quad *$$

$\delta u \rightarrow \boxed{} \rightarrow \delta x$

Example

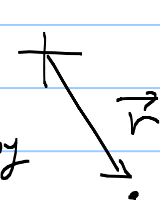


— ②

$$ml^2\ddot{\theta} + l\sin\theta \cdot mg - u = 0$$

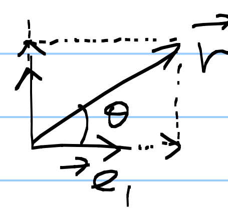
unit vectors $|\vec{e}_j| = 1$

define, $\vec{e}_1\ \vec{e}_2\ \vec{e}_3$ & $\vec{r}$ by



Then $\vec{r} = l\cos\theta\,\vec{e}_1 + l\sin\theta\,\vec{e}_2$ — ①

by



Applied Torque Input

$\vec{T} = u\vec{e}_3$ — ③

From ①, ②, ③ the motion of eq

ml^2: moment of Inertia at the hinge

$$\vec{T} + \vec{r} \times \vec{F}_g = I\ddot{\theta}\vec{e}_3$$

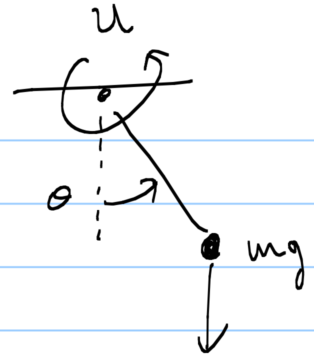
$$u\vec{e}_3 + (l\cos\theta\,\vec{e}_1 + l\sin\theta\,\vec{e}_2) \times (mg\vec{e}_1) = ml^2\ddot{\theta}\vec{e}_3$$

use $\vec{e}_1 \times \vec{e}_1 = \vec{0}$ $\quad \vec{e}_2 \times \vec{e}_1 = -\vec{e}_3$

then $u\vec{e}_3 + l\sin\theta\, mg(-\vec{e}_3) = ml^2\ddot{\theta}\vec{e}_3$

or $(ml^2\ddot{\theta} + l\sin\theta\, mg - u)\,\vec{e}_3 = 0$

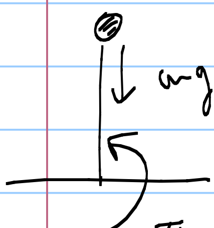
$\therefore \longrightarrow \quad m l^2 \ddot{\theta} + l \sin\theta \, mg - u = 0$



$$m l^2 \ddot{\theta} + l \sin\theta \, mg - u = 0$$

$$\ddot{\theta} + \underbrace{\boxed{\frac{g \sin\theta}{l} - \frac{u}{m l^2}}}_{f(\theta, u)} = 0$$

Linearize at $\underbrace{\theta_0 = \pi}$



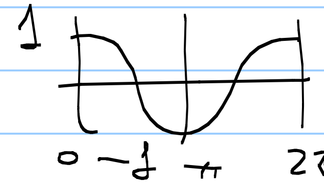
$$\theta = \theta_0 + \delta\theta = \pi + \delta\theta$$

$$\ddot{\theta} = \delta\ddot{\theta}$$

$u = u_0 + \delta u$
$= \delta u$

$\theta = \pi \qquad \overset{0}{\cancel{\ddot{\pi}}} + \overset{0}{\cancel{\frac{\sin\pi \, g}{l}}} - \frac{\overset{u_0}{u_0}}{m l^2} = 0 \longrightarrow \underline{u_0 = 0}$

$$\left\{ \begin{aligned} \left.\frac{\partial f(\theta, u)}{\partial \theta}\right|_{\substack{\theta=\pi \\ u=0}} &= \left.\frac{g\cos\theta}{l}\right|_{\theta=\pi} = -\frac{g}{l} \\ \left.\frac{\partial f(\theta, u)}{\partial u}\right|_{\substack{\theta=\pi \\ u=0}} &= -\frac{1}{m l^2} \end{aligned} \right.$$



$$\delta\ddot{\theta} + \left.\frac{\partial f}{\partial \theta}\right|_{(\pi,0)} \delta\theta + \left.\frac{\partial f}{\partial u}\right|_{(\pi,0)} \delta u = 0$$

$$\delta\ddot{\theta} - \frac{g}{l}\delta\theta - \frac{1}{m l^2}\delta u = 0$$