Quiz 3

A fixed control volume is drawn for *compressible* flow between two stationary, parallel plates. At sections (1) and (2) of the control surface, the x-component of velocity is a function of \( y \) and the density is uniform over each area. The \( y \) and \( z \) velocity components are zero everywhere, so that \( \mathbf{V}_1 = u_1(y) \mathbf{i} \) and \( \mathbf{V}_2 = u_2(y) \mathbf{i} \). Sections (1) and (2) are normal to the direction of the flow. The velocity is zero on the top and bottom of the control surface.

The surface integral that appears in the control volume formulation of the conservation-of-mass equation has the general form:

\[
\text{INT} = \int_C \rho \left( \mathbf{V} \cdot \hat{n} \right) dA
\]

where \( \rho \) is the fluid density, \( \mathbf{V} \) is the fluid velocity vector, \( \hat{n} \) is the unit normal to the control surface, and \( dA \) is a differential element of area on the control surface.

Which of the following versions of the integral are valid? Include a brief explanation for each.

i) \( \text{INT} = \rho \int_C \left( \mathbf{V} \cdot \hat{n} \right) dA \)

ii) \( \text{INT} = \rho_1 \int_{C.S.(1)} \left( \mathbf{V} \cdot \hat{n} \right) dA_1 + \rho_2 \int_{C.S.(2)} \left( \mathbf{V} \cdot \hat{n} \right) dA_2 \)

iii) \( \text{INT} = \int_{C.S.(1)} \rho u \, dA_1 + \int_{C.S.(2)} \rho u \, dA_2 \)

iv) \( \text{INT} = \int_{C.S.(2)} \rho u \, dA_2 - \int_{C.S.(1)} \rho u \, dA_1 \)

v) \( \text{INT} = 2\pi \int_0^h \rho \left( \mathbf{V} \cdot \hat{n} \right)_y \, dy \)
i) Not valid. The flow is compressible and so density may vary over the control surface. Therefore, it cannot be taken outside the integral over the control surface.

ii) Valid. The integral over the control surface has been broken into integrals over sections (1) and (2), as these are the only sections of the control surface where the fluid velocity is non-zero. The density is specified as uniform over the area of each section and so can be pulled out of each integral, recognizing that it may still have a different value at (1) than (2).

iii) Not valid. The integral over the control surface has again been broken into integrals over sections (1) and (2). The dot product \( \mathbf{V} \cdot \hat{n} \) has been replaced with \( u \) in each integral. However over section (1), the surface normal vector, \( \hat{n} \), will be equal to \( -\mathbf{i} \). Therefore \( \mathbf{V} \cdot \hat{n} = -u \) on (1), not \( u \).

iv) Valid. The integral over the control surface has again been broken into integrals over sections (1) and (2). The dot product \( \mathbf{V} \cdot \hat{n} \) has been written, properly, as \( -u \) over section (1) and \( u \) over section (2). The integral over section (2) has been written first so that the more familiar arrangement of terms as \( \dot{m}_{\text{out}} - \dot{m}_{\text{in}} \) is realized, if the velocity profiles in the figure are directionally accurate.

v) Not valid. This integral is what would be obtained over either section (1) or section (2), not the whole control surface, if the figure showed axisymmetric pipe flow, which it does not. The flow is between stationary, parallel plates and a proper integration over the area of (1) or (2) would take the form of a double integral as: \( \int_{-h}^{b} \int_{z_1}^{z_2} \rho (V \cdot \hat{n}) dz dy \).