Problem 1 (50 points):

A laminar oil film (incompressible and Newtonian) drains down a vertical rod. The flow is steady, axisymmetric and has no “swirl.” Some distance down the flow becomes fully developed, so that continuity gives $V = v_z(r) \hat{e}_z$ only. The air offers no resistance to the flow, and the $r$ and $\theta$ momentum equations reduce to $\frac{\partial P}{\partial r} = 0$ and $\frac{\partial P}{\partial \theta} = 0$, respectively.

i) Eliminate terms from the $z$-momentum equation below to obtain a form that can be solved for $v_z(r)$. Note ALL the reasons for which a term could be eliminated.

$$\frac{\partial (\rho v_z)}{\partial t} + v_r \frac{\partial (\rho v_z)}{\partial r} + v_\theta \frac{\partial (\rho v_z)}{\partial \theta} + v_z \frac{\partial (\rho v_z)}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + g_z + \frac{\mu}{\rho} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right]$$

ii) Solve for $v_z(r)$. Clearly state the boundary conditions that you use to do so.

iii) Write, but DO NOT EVALUATE, the integral expressing the volume flow rate, $Q$.

iv) How would you obtain the mean velocity?
After cancelling terms:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) = \frac{1}{\mu} \left( \frac{\partial p}{\partial z} - \rho g \right), \quad \text{using} \quad \rho g = \rho \quad (\text{z-axis points down}) \]

\( p = p(z) \) only, since \( \frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0 \Rightarrow \frac{\partial p}{\partial r} = p'(z) \) only

Since \( \frac{\partial p}{\partial z} |_{r=b} = 0 \), \( \frac{\partial p}{\partial z} = 0 \) everywhere

\[ \frac{\partial}{\partial r} \left( r \frac{\partial v_e}{\partial r} \right) = -\frac{\rho g}{\mu} r, \quad r \frac{\partial v_e}{\partial r} = -\frac{\rho g}{2\mu} r^2 + f(\theta, z) \]

2-D, F.D. flow \( \Rightarrow f(\theta, z) = \text{constant} = c_i \)

\[ \frac{\partial v_z}{\partial r} = -\frac{\rho g}{2\mu} r + \frac{c_i}{r} \]

\( \frac{\partial v_z}{\partial r} \bigg|_{r=b} = 0 = -\frac{\rho g b}{2\mu} + \frac{c_i}{b} \Rightarrow c_i = \frac{\rho gb^2}{2\mu} \)

\[ v_z = -\frac{\rho g}{4\mu} r^2 + c_i \ln r + c_2 \]

\( \text{B.C.:} \ v_z \big|_{r=a} = 0 \)

\[ v_z = -\frac{\rho g}{4\mu} a^2 + c_i \ln a + c_2 \Rightarrow c_2 = -c_i \ln a + \frac{\rho gb^2}{4\mu} a^2 \]

\[ v_z = -\frac{\rho g}{4\mu} r^2 + \frac{\rho gb^2}{2\mu} \ln r - \frac{\rho gb^2}{2\mu} \ln a + \frac{\rho g}{4\mu} a^2 \]

\[ = \frac{\rho g}{2\mu} \left( \frac{a^2 - r^2}{2} + b^2 \ln \left( \frac{r}{a} \right) \right) \]

\[ \text{F.D.} \]

[Diagram of boundary conditions and flow direction]
1) \( \mathbb{Q} \) \( \int_0^b \int_{-\infty}^\infty \phi(x) \, dx \, dy \)

(2v) Divide by the cross-sectional area of the flow, in this case \( \pi (b^2 - a^2) \)
Problem 2 (30 points):

Some cities make use of a combination pump/turbine, as shown in the figure. Water is pumped to the higher reservoir at night, when electricity is cheap. During the day it flows back down through the turbine, providing power when electricity is more expensive. For this system, assume that the head loss is 19 ft (in either direction) at a design condition, where the flow rate is 12,000 gal./min.

i) What is the power requirement (in kW) for the pump?

ii) What will be the power output (in kW) of the turbine?
2) For Pump:
\[ \frac{P_2 - P_1}{\gamma} + \frac{V_2^2 - V_1^2}{2g} + z_2 - z_1 = h_p - \text{loss} \]

\[ P_2 = P_1 = P_{\text{Atm}}, \quad V_2 = V_1 = 0 \quad \text{(LARGE RESERVOIRS)} \]

\[ h_p = z_2 - z_1 + \text{loss} = 125 \text{ ft} + 19 \text{ ft} = 144 \text{ ft} \]

For Turbine! Upper Reservoir is Station 1, Lower is Station 2.

\[ h_T = 125 \text{ ft} + 19 = 144 \text{ ft} \quad \text{WORK OUT OF FLUID} \]

\[ w_{p,T} = gh_{p,T}, \quad w_{f,T} m = gh_{p,T} Q = y h_{p,T} Q \]

\[ W_p = (62.4 \frac{16}{\text{ft}^2})(12,000 \frac{\text{gal}}{\text{min}})(144 \text{ ft})(1 \frac{\text{ft}^3}{7.48 \text{ gal}})(1 \text{ min}) \]

\[ = 2.40 \cdot 10^5 \left( \frac{16}{5} \right) \left( \frac{1 \text{ W}}{1.356 \frac{\text{ft}^2 \cdot 10}{\text{s}}} \right) = 177 \text{ kW} \]

\[ W_T = W_p \frac{h_T}{h_p} = (177 \text{ kW}) \left( \frac{144}{144} \right) = 130 \text{ kW} \]

Since problem asks for power output of turbine, this could have been omitted.