Signals and Systems
Summer 2007

30 May 2007
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1) Representation of DT signals in terms of shifted unit samples
2) Convolution sum representation of DT LTI systems
3) Examples
4) The unit sample response and properties of DT LTI systems
Exploiting Superposition and Time-Invariance

\[ x[n] = \sum_k a_k x_k[n] \xrightarrow{Linear\,System} y[n] = \sum_k a_k y_k[n] \]

**Question:** Are there sets of “basic” signals so that

b) We can represent rich classes of signals as linear combinations of these building block signals.

c) The response of LTI Systems to these basic signals are both *simple* and *insightful*.

**Fact:** For LTI Systems (CT or DT) there are two natural choices for these building blocks

**Focus for now:**

<table>
<thead>
<tr>
<th>Domain</th>
<th>Signals</th>
</tr>
</thead>
<tbody>
<tr>
<td>DT</td>
<td>Shifted unit samples</td>
</tr>
<tr>
<td>CT</td>
<td>Shifted unit impulses</td>
</tr>
</tbody>
</table>
Representation of DT Signals Using Unit Samples

\[ x[-1] + x[0] \delta[n] + x[1] \delta[n-1] + x[-1] \delta[n+1] + x[2] \delta[n-2] \]
That is ...

\[ x[n] = \cdots + x[-2] \delta[n + 2] + x[-1] \delta[n + 1] + x[0] \delta[n] + x[1] \delta[n - 1] + \cdots \]

\[ \downarrow \]

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \]

Coefficients \hspace{2cm} Basic Signals

The Sifting Property of the Unit Sample
Suppose the system is linear, and define $h_k[n]$ as the response to $\delta[n - k]$:

$$\delta[n - k] \rightarrow h_k[n]$$

From superposition:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] h_k[n]$$
Now suppose the system is LTI, and define the unit sample response \( h[n] \):

\[ \delta[n] \rightarrow h[n] \]

From TI:

\[ \delta[n - k] \rightarrow h[n - k] \]

From LTI:

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]
\]

Convolution Sum
Convolution Sum Representation of Response of LTI Systems

\[ y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k] \]

Interpretation

\[ x[k] \delta[n - k] \]

\[ \delta[n] \]

\[ \rightarrow \]

\[ h[n] \]

\[ \uparrow \]

\[ \rightarrow \]

\[ x[k] h[n - k] \]

Sum up responses over all \( k \)
Visualizing the calculation of $y[n] = x[n] * h[n]$

Choose value of $n$ and consider it fixed

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

View as functions of $k$ with $n$ fixed
Calculating Successive Values: Shift, Multiply, Sum

\[ y[n] = 0 \quad \text{for } n < \]
\[ y[-1] = \]
\[ y[0] = \]
\[ y[1] = \]
\[ y[2] = \]
\[ y[3] = \]
\[ y[4] = \]
\[ y[n] = 0 \quad \text{for } n > \]
Properties of Convolution and DT LTI Systems

1) A DT LTI System is *completely characterized* by its unit sample response

Ex. #1: \( h[n] = \delta[n - n_0] \)

There are *many* systems with this response to \( \delta[n] \)

There is only *one* LTI System with this response to \( \delta[n] \):

\[
y[n] = x[n - n_0]
\]

\[\downarrow\]

\[
x[n] * \delta[n - n_0] = x[n - n_0]
\]
Ex. #2:

\[ y[n] = \sum_{k=-\infty}^{n} x[k] \quad \text{- An Accumulator} \]

Unit Sample response

\[ h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n] \]

\[ x[n] \ast u[n] = \sum_{k=-\infty}^{n} x[k] \]
The Commutative Property

\[ y[n] = x[n] \ast h[n] = h[n] \ast x[n] \]

Ex: Step response \( s[n] \) of an LTI system

\[ s[n] = u[n] \ast h[n] = h[n] \ast u[n] \]

\[ s[n] = \sum_{k=-\infty}^{n} h[k] \]
The Distributive Property

\[ x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] \]

Interpretation

\[ x[n] \rightarrow h_1[n] + h_2[n] \rightarrow y[n] \]

\[ \begin{array}{c}
\vdots \\
\end{array} \]

\[ x[n] \rightarrow h_1[n] \rightarrow \]

\[ + \]

\[ x[n] \rightarrow h_2[n] \rightarrow y[n] \]
The Associative Property

\[ x[n] \ast (h_1[n] \ast h_2[n]) = (x[n] \ast h_1[n]) \ast h_2[n] \]

(Commutativity)

\[ x[n] \ast (h_2[n] \ast h_1[n]) = (x[n] \ast h_2[n]) \ast h_1[n] \]

Implication (Very special to LTI Systems)
Properties of LTI Systems

1) Causality ⇔ $h[n] = 0$ for all $n < 0$

2) Stability ⇔ $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$