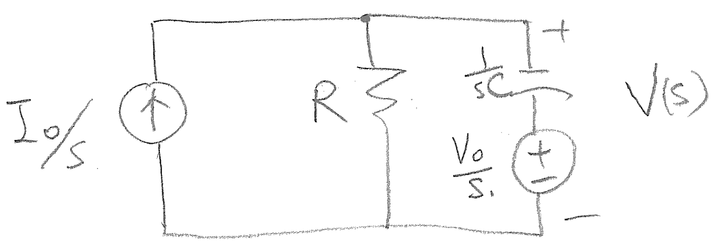
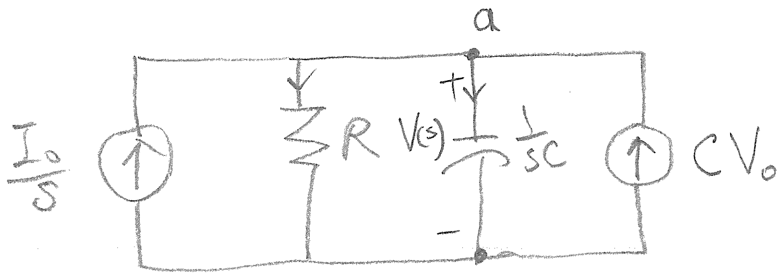
(P.1) [13] For the capacitor, $v(0^-) = V_0$.

The circuit diagram becomes



With a source transformation, the circuit becomes



Applying KCL to node a yields

$$\frac{I_0}{S} - \frac{V(s)}{R} - \frac{V_s}{\frac{1}{sC}} + CV_0 = 0$$

$$\therefore \quad \frac{I_0}{S} + CV_0 = \frac{V(s)}{R} + V(s)\,sC = \left(sC + \frac{1}{R}\right)V(s)$$

$$V(s) = \frac{I_0}{S\left(sC + \frac{1}{R}\right)} + \frac{CV_0}{\left(sC + \frac{1}{R}\right)}$$

$$= \frac{V_0}{S + \frac{1}{RC}} + \frac{I_0/C}{S\left(S + \frac{1}{RC}\right)}$$

$$= \frac{V_0}{S + \frac{1}{RC}} + \frac{k_1}{S} + \frac{k_2}{S + \frac{1}{RC}}$$

$$k_1 = \frac{I_0/C}{0+\frac{1}{RC}} = I_0 R$$

$$k_2 = \left(S+\frac{1}{RC}\right)\frac{I_0/C}{S\left(S+\frac{1}{RC}\right)}\Bigg|_{S=-\frac{1}{RC}} = \frac{I_0/C}{-\frac{1}{RC}} = -I_0 R$$

$$\therefore \; V(s) = \frac{V_0}{S+\frac{1}{RC}} + \frac{I_0 R}{S} - \frac{I_0 R}{S+\frac{1}{RC}}$$

$$= \left(V_0 - I_0 R\right)\frac{1}{S+\frac{1}{RC}} + I_0 R\,\frac{1}{S}$$

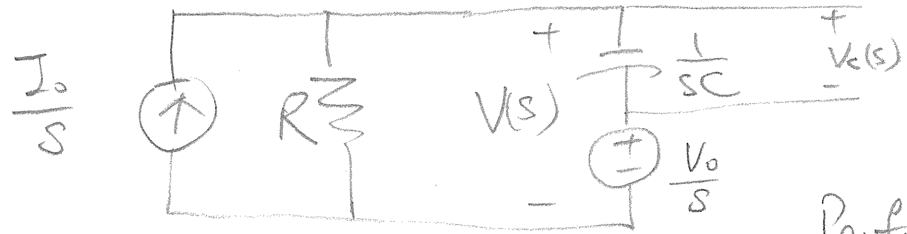
$$\boxed{\therefore \; v(t) = \left[\left(V_0 - I_0 R\right)e^{-\frac{t}{RC}} + I_0 R\right]u(t) \;\; V}$$

2 2 2 2 2 1

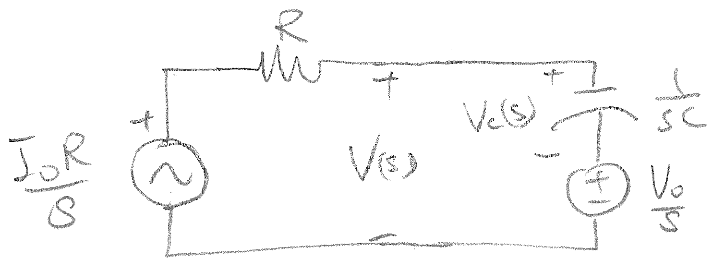
$$\boxed{\tau = RC}$$

2

(P.1) Alternate Approach



Perform a source transformation ...



$$V_{c_1}(s) = \frac{I_0 R}{s} \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{I_0 R}{s} \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$V_{c_2}(s) = -\frac{V_0}{s} \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = -\frac{V_0}{s} \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$V_c(s) = \frac{(I_0 R - V_0)\frac{1}{RC}}{s\left(s + \frac{1}{RC}\right)} = \frac{k_1}{s} + \frac{k_2}{s + \frac{1}{RC}}$$

$$k_1 = I_0 R - V_0$$

$$k_2 = \frac{(I_0 R - V_0)\frac{1}{RC}}{-\frac{1}{RC}} = -(I_0 R - V_0)$$

$$\therefore V_c(s) = -(V_0 - I_0 R)\frac{1}{s} + (V_0 - I_0 R)\frac{1}{s + \frac{1}{RC}}$$

$$V(s) = (V_0 - I_0 R)\frac{1}{s + \frac{1}{RC}} + I_0 R \frac{1}{s}. = V_c(s) + \frac{V_0}{s}$$

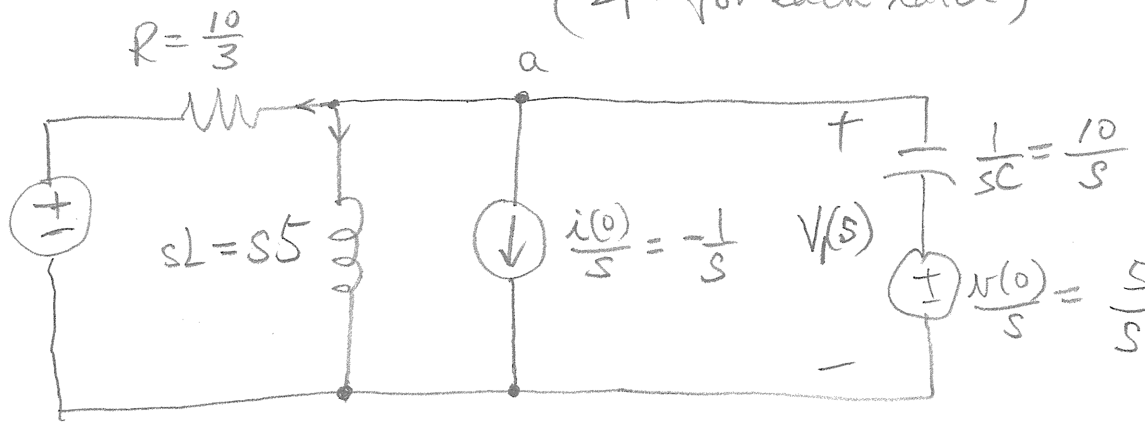
$$\therefore \boxed{v(t) = \left[(V_0 - I_0 R)e^{-\frac{t}{RC}} + I_0 R\right] u(t) \ V}$$

$$\boxed{\text{time constant } \tau = RC.}$$

(P.2) [50] (2pts for each label) 4/10

(a) [14]

$V_s(s) = \frac{10}{s}$



(b) [11]

Apply KCL at node a yields:

$$\frac{V_1(s) - 10/s}{\frac{10}{3}} + \frac{V_1(s)}{s5} - \frac{1}{s} + \frac{V_1(s) - 5/s}{10/s} = 0.$$

$$(0.3)V_1(s) - \frac{3}{s} + \frac{V_1(s)}{s5} - \frac{1}{s} + \frac{sV_1(s)}{10} - \frac{1}{2} = 0.$$

$$V_1(s)\left[0.3 + \frac{1}{s}0.2 + 0.1\,s\right] = \frac{3}{s} + \frac{1}{s} + 0.5$$

$$\frac{0.1}{s}V_1(s)\left[3s + 2 + s^2\right] = \frac{4}{s} + 0.5$$

$$V_1(s)\left[s^2 + 3s + 2\right] = 40 + 5s$$

$$\therefore \quad V_1(s) = \frac{5s+40}{s^2+3s+2} = \frac{5s+40}{(s+2)(s+1)}$$

$$= \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$k_1 = \frac{35}{1} = 35$$

$$k_2 = \frac{30}{-1} = -30$$

$$\therefore \quad V_1(s) = \frac{35}{s+1} - \frac{30}{s+2}$$

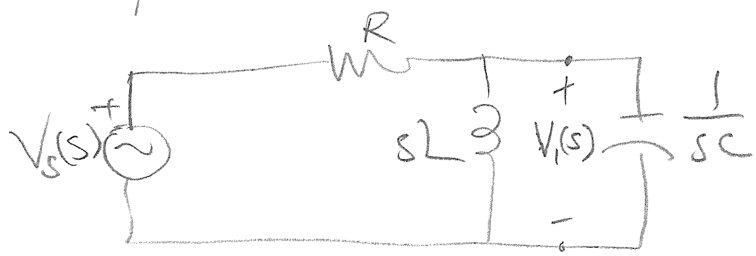
thus

(b) $$\boxed{v_1(t) = \left[35e^{-t} - 30e^{-2t}\right] u(t) \ \ V}$$

2 2 2 2 2 1

P. 2 (c) [5]

To obtain the network function, set all ICs to 0.



$$V_1(s) = V_s(s) \frac{sL // \frac{1}{sC}}{sL // \frac{1}{sC} + R}$$

note:

$$sL // \frac{1}{sC} = \frac{sL \frac{1}{sC}}{sL + \frac{1}{sC}} = \frac{sL}{s^2LC+1}$$

$$T_1(s) = \frac{V_1(s)}{V_s(s)} = \frac{\frac{sL}{s^2LC+1}}{\frac{sL}{s^2LC+1} + R} = \frac{\frac{sL}{s^2LC+1}}{\frac{sL + Rs^2LC + R}{s^2LC+1}}$$

$$= \frac{sL}{s^2RLC + sL + R}$$

$$= \frac{s\frac{1}{RC}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}}$$

note: $\frac{1}{RC} = \frac{1}{\left(\frac{10}{3}\right)(0.1)} = 3$

and $\frac{1}{LC} = \frac{1}{5(0.1)} = 2.$

(c) $$\boxed{T_1(s) = \frac{3s}{s^2+3s+2}}$$

2 3

$$H(s) = 1 \cdot T_1(s) = \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

P.2(d)
[10]

$$k_1 = \left.\frac{3s}{s+1}\right|_{s=-2} = \frac{-6}{-1} = 6$$

$$k_2 = \left.\frac{3s}{s+2}\right|_{s=-1} = \frac{-3}{1} = -3$$

$$\therefore \quad H_1(s) = \frac{6}{s+2} - \frac{3}{s+1}$$

(d) $$\boxed{h_1(t) = \left[6e^{-2t} - 3e^{-t}\right] u(t)}$$

2 2 2 2 2

P.2(e) Step response:
[10]

To obtain the step response, $g(t)$, we use the relationship:

$$G(s) = \frac{H_1(s)}{s} = \frac{3}{s^2+3s+2} = \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

$$k_1 = \left.\frac{3}{s+1}\right|_{s=-2} = -3$$

$$k_2 = \left.\frac{3}{s+2}\right|_{s=-1} = 3$$

$$\therefore G(s) = \frac{-3}{s+2} + \frac{3}{s+1}$$

$\therefore$ (e) $$\boxed{g(t) = \left[-3e^{-2t} + 3e^{-t}\right] u(t)}$$

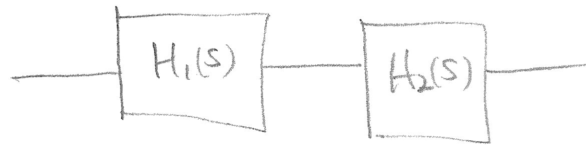
2 2 2 2 2

7/10

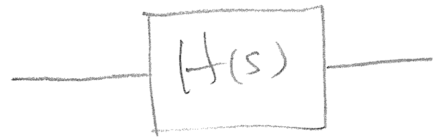
P.3 [18]

$h_1(t) = e^{-2t}$

$h_2(t) = 4e^{-4t}$



The cascaded system is



$H(s) = H_1(s) \cdot H_2(s)$

which has the time-domain relation

$h(t) = h_1(t) * h_2(t)$

(i) [10] The time-domain integral is $h(t) = \int_0^t h_1(t-\tau)\, h_2(\tau)\, d\tau$

$$h(t) = \int_0^t e^{-2(t-\tau)} \cdot 4e^{-4\tau}\, d\tau \quad 2$$

$$= \int_0^t 4e^{-2t} e^{-2\tau}\, d\tau$$

$$= 4e^{-2t} \int_0^t e^{-2\tau}\, d\tau$$

$$= \left(4e^{-2t}\right)\left(-\frac{1}{2}\right) e^{-2\tau}\Big|_0^t$$

$$= -2e^{-2t}\left[e^{-2t} - 1\right]$$

$$\therefore \boxed{h(t) = \left(2e^{-2t} - 2e^{-4t}\right)} u(t)$$

2 2 2 2

8/10

P. 3 (ii)

[8]

$$H(s) = H_1(s) \cdot H_2(s)$$

$$H_1(s) = \mathcal{L}\{h_1(t)\} = \frac{1}{s+2}$$

$$H_2(s) = \mathcal{L}\{h_2(t)\} = \frac{4}{s+4}$$

$$\therefore \quad H(s) = \frac{1}{s+2} \cdot \frac{4}{s+4}$$

$$= \frac{k_1}{s+2} + \frac{k_2}{s+4}$$

$$k_1 = \left.\frac{4}{s+4}\right|_{s=-2} = 2$$

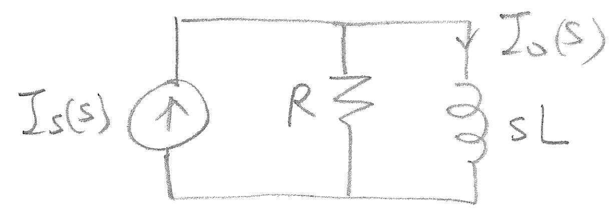
$$k_2 = \left.\frac{4}{s+2}\right|_{s=-4} = -2$$

$$\therefore \quad H(s) = 2\frac{1}{s+2} - \frac{2}{s+4}$$

$$\therefore \quad \boxed{h(t) = \left(2e^{-2t} - 2e^{-4t}\right)} u(t)$$

2 2 2 2

P.4
[19]



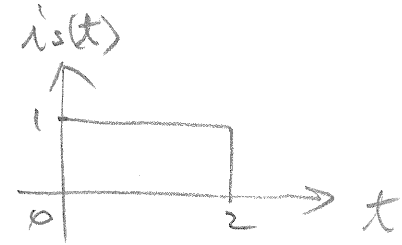
$R = 1\,\Omega$
$L = 1H$

To use the convolution integral, we need the impulse response, $h(t)$, of the circuit.
$h(t)$ may be obtained as $\mathcal{L}^{-1}\{H(s)\}$.
The transfer function, $H(s)$, of the circuit is

$$H(s) = \frac{I_o(s)}{I_s(s)} = \frac{R}{R+sL} = \frac{1}{s+1}$$

$$\therefore\ h(t) = e^{-t}\,u(t)$$

The input signal is



$$i_s(t) = u(t) - u(t-2)$$

Thus, the output is

$$i_o(t) = h(t) * i_s(t)$$

$$= \int_0^t i_s(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_0^t [u(\tau) - u(\tau-2)]\,e^{-(t-\tau)}\,d\tau$$

P. 4 (cont'd)

Since $u(\tau-2)=0$ for $0<\tau<2$.

the integrand involving $u(\tau)$ is nonzero $\forall\ \tau>0$.

The integrand involving $u(\tau-2)$ is nonzero only for $\lambda>2$

Thus, the integration / convolution will be performed in 2 parts, with

$$i_o(t) = i_1(t) + i_2(t)$$

$$i_1(t) = \int_0^t (1) e^{-(t-\lambda)} d\tau$$

$$= e^{-t}\int_0^t (1) e^{\tau} d\tau$$

$$= e^{-t} e^{\tau}\Big|_0^t = e^{-t}\left[e^{t}-1\right]$$

$$i_1(t) = 1-e^{-t} \quad t\geq 0$$

And

$$i_2(t) = \int_2^t (1) e^{-(t-\tau)} d\tau = e^{-t}\int_2^t e^{\tau} d\tau$$

$$= e^{-t}\left[e^{t}-e^{2}\right]$$

$$= 1-e^{2}e^{-t} = 1-e^{-t+2} \quad \text{for } t\geq 2$$

$\therefore$ $\boxed{i_o(t) = \underset{1}{}\left(1-e^{-t}\right)_{2\ 2}\left[u(t)\right]_{2} \underset{2}{-} \left(1-e^{-(t-2)}\right)_{2\ 2\ 2} u(t-2)_{2}\ A_{2}}$

OR $\boxed{i_o(t) = \begin{cases} 1-e^{-t} & A \quad 0\leq t\leq 2 \\ (e^{2}-1)e^{-t} & A \quad t>2 \end{cases}}$ 2, 2, 1, 2, 2 / 2, 2, 2, 1, 2

$\Downarrow$

supplemental

10/10

Are the two forms of solution really equal to each other?

$$i_o(t) = (1-e^{-t})u(t) - \left[1-e^{-(t-2)}\right]u(t-2) \quad \text{— (1)}$$

$$i_o(t) = \begin{cases} 0 & t<0 \\ 1-e^{-t} & 0 \leq t \leq 2 \\ (e^2-1)e^{-t} & 2<t \end{cases} \quad \text{— (2)}$$

(a) For $t<0$, (1) yields zero ✓

(b) For $0 \leq t \leq 2$:
the 2nd term in (1) is zero due to $u(t-2)$.
The output is due to contribution from the 1st term in (1):

$$i_o(t) = (1-e^{-t})\cdot 1 - \left[1-e^{-(t-2)}\right]\cdot 0$$

$$i_o(t) = 1-e^{-t} \quad 0 \leq t \leq 2 \quad ✓$$

(c) For $t>2$:
Both terms in (1) contribute to the answer:

$$i_o(t) = (1-e^{-t})\cdot 1 - (1-e^{-(t-2)})\cdot 1 \quad \text{for } t>2$$

$$= -e^{-t} + e^{-t}e^{2}$$

$$i_o(t) = e^{-t}(e^2-1) \quad t>2 \quad ✓$$

$\therefore$ Yes, the two forms, (1) & (2) are equal!