Optimal Multiband Transmission Under Hostile Jamming

Tianlong Song, Wayne E. Stark, Tongtong Li, and Jitendra K. Tugnait

***Abstract*—This paper considers optimal multiband transmission under hostile jamming, where both the authorized user and the jammer are power-limited and operate against each other. The strategic decision-making of the authorized user and the jammer is modeled as a two-party zero-sum game, where the payoff function is the capacity that can be achieved by the authorized user in presence of the jammer. *First*, we investigate the game under AWGN channels. It is found that: either for the authorized user to maximize its capacity, or for the jammer to minimize the capacity of the authorized user, the best strategy for both of them is to distribute the transmission power or jamming power uniformly over all the available spectrum. The minimax capacity can be calculated based on the channel bandwidth and the signal-to-jamming and noise ratio, and it matches with the Shannon channel capacity formula. *Second*, we consider frequency selective fading channels. We characterize the dynamic relationship between the optimal signal power allocation and the optimal jamming power allocation in the minimax game, and then propose an iterative water pouring algorithm to find the optimal power allocation schemes for both the authorized user and the jammer.**

***Index Terms*—Multiband transmission, jamming, capacity analysis, game theory.**

I. Introduction

HOSTILE jamming, in which the authorized user's signal is deliberately interfered by the adversary, is one of the most commonly used techniques for limiting the effectiveness of an opponent's communication [1]. In traditional research on jamming strategy and jamming mitigation, there is generally an assumption that the jammer or the authorized user can access at least part of the information about the transmission pattern of its adversary. As such, the jammer can launch more effective jamming by exploiting the information it has about the authorized user, e.g., correlated jamming [2]–[4] or disguised jamming [5]–[9]. For jamming mitigation, the authorized user can mitigate the jammer's effect by applying a particular anti-jamming scheme that is robust against a specific jamming pattern [10], [11]. The underlying assumption is that the jamming varies slowly such that the authorized user has sufficient time to track and react to the jamming.

However, if the jammer is intelligent and can switch its patterns fast enough, then it would be impossible for the authorized user to track and react in real time. In this case, when choosing the strategy to maximize its capacity, the authorized user has no knowledge of the jamming strategy. Similarly, while trying to minimize the capacity of the authorized user, the jammer has no knowledge of the user strategy, either. Regarding this scenario, there has been a surge in research that applies game theory to characterize and analyze strategies for communication systems under jamming with unpredictable strategies.

A lot of work on game theory in communications has been focused on *the single user and single band case* [12]–[16]. The optimal jamming strategy under the Gaussian test channel was investigated in [12], and the worst additive noise for a communication channel under a covariance constraint was studied in [13]. The capacity of channels with block memory was investigated in [14], which showed that both the optimal coding strategy and the optimal jamming strategy are independent from symbol to symbol within a block. The authors in [15] discussed the minimax game between an authorized user and a jammer for any combinations of "hard" or "soft" input and output quantization with additive noise and average power constraints. In [16], a dynamic game between a communicator and a jammer was considered, where the participants choose their power levels randomly from a finite space subject to temporal energy constraints.

Application of game theory to *multiuser and multiband/multicarrier communications* has been brought to attention in recent years [17]–[21]. In [17], the authors proposed a decentralized strategy to find out the optimal precoding/multiplexing matrices for a multipoint-to-multipoint communication system composed of a set of wideband links sharing the same physical resources. In [18], a scheme aiming for fair allocation of subcarriers, rates, and power for multiuser orthogonal frequency-division multiple-access (OFDMA) systems was proposed to maximize the overall system rate, subject to each user's maximal power and minimal rate constraints. In [19], jamming mitigation was carried out by maximizing the sum signal-to-interference and noise ratio (SINR) for multichannel communications. In [20], the authors considered a particular scenario where K users and a jammer share a common spectrum of N orthogonal tones, and examined how each user could maximize its own total sum rate selfishly. In [21], the authors investigated the secrecy capacity of the users under malicious eavesdropping and friendly jamming.

Game theory has also been applied to *cognitive radios and*

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ad hoc networks [22]–[26]. New techniques for analyzing networks of cognitive radios that can alter either their power levels or their signature waveforms through the use of game models were introduced in [22]. In [23], a game theoretic overview of dynamic spectrum sharing was provided regarding analysis of network users' behaviors, efficient dynamic distributed design, and performance optimality. A game theoretic power control framework for spectrum sensing in cognitive radio networks was proposed in [24], and the minimax game for cooperative spectrum sensing in centralized cognitive radio networks was investigated in [25]. In [26], the authors developed a game theoretic framework to construct convergent interference avoidance (IA) algorithms in ad hoc networks with multiple distributed receivers.

For spectrum and power utilization in multiband communications, an open while interesting question is: in presence of a random and intractable opponent, can the authorized user or the jammer benefit from utilizing part instead of the entire spectrum and/or applying nonuniform power allocation?

In this paper, we try to address this question from a game theoretic perspective, taking jamming and jamming mitigation as a game between a power-limited jammer and a power-limited authorized user, who operate against each other over the same spectrum consisting of multiple bands or subchannels. The authorized user is always trying to maximize its capacity under jamming by applying an optimal strategy. Accordingly, the jammer would like to find an optimal strategy that can minimize the capacity of the authorized user. To apply a chosen strategy, the authorized user or the jammer selects a particular number of subchannels and applies a particular power allocation scheme over the selected subchannels. For both the authorized user and the jammer, the subchannels may not be chosen with equal probability. The strategic decision-making of the authorized user and the jammer can be modeled as a two-party zero-sum game, where the payoff function is the capacity that can be achieved by the authorized user in presence of the jammer.

Solving the zero-sum game above is equivalent to locating the saddle point, which produces optimal strategies for both the authorized user and the jammer. That is, the jammer cannot reduce the capacity of the authorized user by applying a jamming strategy different from the optimal one; meanwhile, the authorized user cannot increase its capacity by switching to another transmission strategy either. We find that: under AWGN channels, either for the authorized user to maximize its capacity, or for the jammer to minimize the capacity of the authorized user, the best strategy for both of them is to distribute the signal power or jamming power uniformly over all the available spectrum. The minimax capacity of the authorized user is given by $C = B\log_2(1+\frac{P_s}{P_J+P_N})$, where B is the bandwidth of the overall spectrum, P_N the noise power, P_s and P_J the total power of the authorized user and the jammer, respectively. In other words, the minimax capacity above is the minimal capacity that can be achieved by the authorized user if it utilizes all the available spectrum and applies uniform power allocation, no matter what strategy is applied by the jammer; meanwhile, it is also the maximal capacity that can be achieved by the authorized user if the jammer jams all the available spectrum and applies uniform power allocation, no matter what strategy is applied by the authorized user.

As can be expected, the results we obtained under AWGN channels may no longer be true for frequency selective fading channels. In the jamming-free case, it is well known that the classical water pouring algorithm provides the optimal power allocation scheme that maximizes the capacity of the authorized user under frequency selective fading channels. Naturally, the situation becomes complicated when a jammer is involved in the game.

To identify the saddle point under frequency selective fading channels, we *first* characterize the dynamic relationship between the optimal signal power allocation and the optimal jamming power allocation in the minimax game. *Second*, we show that for correlated fading channels, the closed-form solution for the saddle point can be obtained using a two-step water pouring algorithm. As a special case, it is shown that when the channel for the authorized user and the channel for the jammer are relatively flat with respect to each other, i.e., their magnitude spectrum is proportional to each other, the closed-form solution for the saddle point can be obtained. From the arbitrarily varying channel (AVC) point of view, the correlation between the user channel and the jamming channel can be regarded as an indicator of possible symmetricity between the user and the jammer. It is also observed that as long as the cross-correlation between the user channel and the jammer channel is reasonably high, the two-step water pouring algorithm can still provide a much better solution than uniform power allocation. *Third*, we extend the two-step approach to an iterative water pouring algorithm. The iterative algorithm can find a numerical solution to the saddle point for arbitrary fading channels. It is observed that this algorithm delivers a solution that has a notable advantage over uniform power allocation for both the authorized user and the jammer. Simulation examples are provided to illustrate our findings for both the AWGN channels and the frequency selective fading channels.

The rest of the paper is organized as follows. In Section II, the problem is formulated after the system model description. The minimax problem in the zero-sum game with an authorized user and a jammer under AWGN channels is theoretically solved in Section III. The gaming problem under frequency selective fading channels is investigated in Section IV. Numerical analysis is provided in Section V and we conclude in Section VI.

II. Problem Formulation

A. System Description

We consider a multiband communication system[1], where there is *an authorized user* and *a jammer* who are operating against each other, without knowledge on the strategy applied by its opponent. Assuming that both the authorized user and the jammer can choose to operate over all or part of the

[1]We assume multiband communications here, but the derivation in this paper is readily applicable to multicarrier communications (e.g., OFDM), if the authorized user and the jammer apply the same transceiver structure.

N_c frequency bands or subchannels (not necessarily being consecutive), each of which has a bandwidth $\frac{B}{N_c}$ Hz. We start with the AWGN channel model, where all the subchannels have equal noise power, and then extend to the frequency selective fading scenario. In the AWGN case, assuming the total noise power over the entire spectrum is P_N, then the noise power corresponding to each subchannel is $\frac{P_N}{N_c}$. We assume the jamming is Gaussian over each jammed subchannel, because Gaussian jamming is the worst jamming when the jammer has no knowledge of the authorized transmission [12]. In the following, let P_s denote the total signal power for the authorized user, and P_J the total jamming power.

The authorized user is always trying to maximize its capacity under jamming by applying an optimal strategy on subchannel selection (either all or part) and power allocation. On the other hand, the jammer tries to find an optimal strategy that can minimize the capacity of the authorized user. In this paper, we consider the case where both the authorized user and the jammer use mixed strategies. It is assumed that both the authorized user and the jammer can adjust their subchannel selection and power allocation swiftly and randomly, such that neither of them has sufficient time to learn and react in real time before its opponent switches to new subchannels and/or power levels. In other words, when the authorized user and the jammer apply their own resource allocation strategy, they have no knowledge of the selected subchannels and power levels of their opponent.

B. Strategy Spaces for the Authorized User and the Jammer

Each mixed strategy applied by the authorized user is determined by the number of activated subchannels, the subchannel selection process and the power allocation process. More specifically: (1) The authorized user activates K_s $(1 \leq K_s \leq N_c)$ out of N_c subchannels each time for information transmission. (2) The subchannel selection process is characterized using a binary indicator vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_{N_c}]$, where each random variable $\alpha_m = 1$ or 0 indicates whether the mth subchannel is selected or not, and $\sum_{m=1}^{N_c} \alpha_m = K_s$. Let $\boldsymbol{\omega}_s = [\omega_{s,1}, \omega_{s,2}, ..., \omega_{s,N_c}]$ be the corresponding probability vector, where $\omega_{s,m}$ denotes the probability that the mth subchannel is selected each time. That is, $\omega_{s,m} = Pr\{\alpha_m = 1\}$, and $\sum_{m=1}^{N_c} \omega_{s,m} = K_s$. (A simple strategy for selecting a particular number of subchannels based on a given subchannel selection probability vector, $\boldsymbol{\omega}_s$, is illustrated in Appendix A.) (3) For notation simplicity, the authorized user always specifies the indices of the selected K_s subchannels as $1, 2, ..., K_s$, following the order as they appear in the original spectrum, and performs power allocation over them. The power allocation process is characterized using a vector $\mathbf{P}_s = [P_{s,1}, P_{s,2}, ..., P_{s,K_s}]$, in which $P_{s,n}$ denotes the power allocated to the nth selected subchannel, and $\sum_{n=1}^{K_s} P_{s,n} = P_s$ is the power constraint. Let $\mathcal{W}_{s,K_s} = \{\boldsymbol{\omega}_s = [\omega_{s,1}, \omega_{s,2}, ..., \omega_{s,N_c}] \mid 0 \leq \omega_{s,m} \leq 1, \sum_{m=1}^{N_c} \omega_{s,m} = K_s\}$, and $\mathcal{P}_{s,K_s} = \{\mathbf{P}_s = [P_{s,1}, P_{s,2}, ..., P_{s,K_s}] \mid 0 < P_{s,n} \leq P_s, \sum_{n=1}^{K_s} P_{s,n} = P_s\}$. The strategy space for the authorized user can thus be defined as

$$\mathcal{X} = \{(K_s, \boldsymbol{\omega}_s, \mathbf{P}_s) \mid 1 \leq K_s \leq N_c, \boldsymbol{\omega}_s \in \mathcal{W}_{s,K_s}, \mathbf{P}_s \in \mathcal{P}_{s,K_s}\}. \tag{1}$$

The strategy space $\mathcal{X}$ covers all the possible subchannel utilization strategies as K_s varies from 1 to N_c. Here, a strategy $(K_s, \boldsymbol{\omega}_s, \mathbf{P}_s)$ with $K_s = 1$ and $\boldsymbol{\omega}_s = [\frac{1}{N_c}, \cdots, \frac{1}{N_c}]$ and $P_{s,1} = P_s$, corresponds to the conventional frequency hopping (FH) system, while a strategy $(K_s, \boldsymbol{\omega}_s, \mathbf{P}_s)$ with $K_s = N_c$, $\boldsymbol{\omega}_s = [1, \cdots, 1]$ and $P_{s,n} = \frac{P_s}{N_c}$, $\forall n$, would result in a full band transmission with uniform power allocation.

Similarly, the jammer jams K_J $(1 \leq K_J \leq N_c)$ out of N_c subchannels each time following a binary indicator vector $\boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_{N_c}]$ with $\sum_{m=1}^{N_c} \beta_m = K_J$. The subchannel selection process is characterized using a probability vector $\boldsymbol{\omega}_J = [\omega_{J,1}, \omega_{J,2}, ..., \omega_{J,N_c}]$, where $\omega_{J,m} = Pr\{\beta_m = 1\}$ and $\sum_{m=1}^{N_c} \omega_{J,m} = K_J$. Then the jammer specifies the indices of the K_J jammed subchannels as $1, 2, ..., K_J$ in the same manner as the authorized user, and performs power allocation over them using a power-allocation vector $\mathbf{P}_J = [P_{J,1}, P_{J,2}, ..., P_{J,K_J}]$ constrained by $\sum_{n=1}^{K_J} P_{J,n} = P_J$. Let $\mathcal{W}_{J,K_J} = \{\boldsymbol{\omega}_J = [\omega_{J,1}, \omega_{J,2}, ..., \omega_{J,N_c}] \mid 0 \leq \omega_{J,m} \leq 1, \sum_{m=1}^{N_c} \omega_{J,m} = K_J\}$ and $\mathcal{P}_{J,K_J} = \{\mathbf{P}_J = [P_{J,1}, P_{J,2}, ..., P_{J,K_J}] \mid 0 < P_{J,n} \leq P_J, \sum_{n=1}^{K_J} P_{J,n} = P_J\}$, the strategy space for the jammer can thus be defined as

$$\mathcal{Y} = \{(K_J, \boldsymbol{\omega}_J, \mathbf{P}_J) \mid 1 \leq K_J \leq N_c, \boldsymbol{\omega}_J \in \mathcal{W}_{J,K_J}, \mathbf{P}_J \in \mathcal{P}_{J,K_J}\}. \tag{2}$$

C. The Minimax Problem in the Zero-Sum Game between the Authorized User and the Jammer

From a game theoretic perspective, the strategic decision-making of the authorized user and the jammer can be modeled as a two-party zero-sum game [27], which is characterized by a triplet $(\mathcal{X}, \mathcal{Y}, C)$, where

1) $\mathcal{X}$ is the strategy space of the authorized user;
2) $\mathcal{Y}$ is the strategy space of the jammer;
3) C is a real-valued payoff function defined on $\mathcal{X} \times \mathcal{Y}$.

The interpretation is as follows. Let (x, y) denote the strategy pair, in which $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are the strategies applied by the authorized user and the jammer, respectively. Note that both x and y are mixed strategies. The payoff function $C(x, y)$ is therefore defined as the *ergodic* (i.e., expected or average) capacity of the authorized user choosing a strategy $x \in \mathcal{X}$ in presence of the jammer choosing a strategy $y \in \mathcal{Y}$. In other words, $C(x, y)$ is the amount that the authorized user wins and simultaneously the jammer loses in the game with a strategy pair (x, y).

Assuming that with strategy pair (x, y), the authorized user and the jammer activate K_s and K_J channels, respectively. Define $\mathcal{A}_{K_s} = \{\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_{N_c}] \mid \alpha_m \in \{0, 1\}, \sum_{m=1}^{N_c} \alpha_m = K_s\}$, and $\mathcal{B}_{K_J} = \{\boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_{N_c}] \mid \beta_m \in \{0, 1\}, \sum_{m=1}^{N_c} \beta_m = K_J\}$. Let $p(\boldsymbol{\alpha}|x)$ denote the probability that the subchannels selected by the authorized user follow the indicator vector $\boldsymbol{\alpha}$ given that the strategy $x \in \mathcal{X}$ is applied, and $p(\boldsymbol{\beta}|y)$ the probability that the subchannels selected by the jammer follow the indicator vector $\boldsymbol{\beta}$ given that the strategy $y \in \mathcal{Y}$ is applied. Let $T_{s,m}$

and $T_{J,m}$ be the power allocated to the mth subchannel by the authorized user and the jammer, respectively, which are determined by

$$T_{s,m} = \begin{cases} P_{s,g_m}, & \alpha_m = 1, \\ 0, & \alpha_m = 0, \end{cases} \quad T_{J,m} = \begin{cases} P_{J,q_m}, & \beta_m = 1, \\ 0, & \beta_m = 0, \end{cases} \tag{3}$$

where $g_m = \sum_{i=1}^{m} \alpha_i$ is the new index of subchannel m specified by the authorized user in the K_s selected subchannels if it is activated by the authorized user ($\alpha_m = 1$), and $q_m = \sum_{i=1}^{m} \beta_i$ is the new index of subchannel m specified by the jammer in the K_J jammed subchannels if it is activated by the jammer ($\beta_m = 1$). Note that: (i) the subchannel selection processes used by the authorized user and the jammer are independent of each other; and (ii) for each strategy pair (x, y), the subchannel selection choices ($\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$) are not unique for both the authorized user and the jammer. Thus, the ergodic capacity of the authorized user in the game with a strategy pair (x, y) can be calculated as

$$\begin{aligned} C(x,y) = &\sum_{\boldsymbol{\alpha} \in \mathcal{A}_{K_s}} \sum_{\boldsymbol{\beta} \in \mathcal{B}_{K_J}} p(\boldsymbol{\alpha}|x) p(\boldsymbol{\beta}|y) \\ &\times \sum_{m=1}^{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{T_{s,m}}{T_{J,m} + P_N/N_c} \right). \end{aligned} \tag{4}$$

In the jamming-free case, the traditional Shannon channel capacity is obtained by maximizing the mutual information with respect to the distribution of the user signal. When jamming is around, the user still wants to maximize its capacity, while the jammer tries to minimize the user's capacity. This is why the "minimax" capacity was introduced in literature [28]–[30], for which the mutual information is *maximized* with respect to the distribution of the user signal, and meanwhile *minimized* with respect to the distribution of the jamming. Now, in addition to optimizing the signal distribution, both the authorized user and the jammer can also choose which subchannels to use or activate, and how much power to be allocated to each activated subchannel. That is, the minimax capacity is obtained through the optimization with respect to the strategies from both the authorized user and the jammer sides, in addition to the signal distributions.

Based on the definitions and reasoning above, the minimax capacity of the authorized user is defined as

$$C(x^*, y^*) = \max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} C(x,y) = \min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} C(x,y). \tag{5}$$

It can be seen from (5) that the authorized user tries to choose an optimal transmission strategy $x^* \in \mathcal{X}$ to maximize its capacity, while the jammer tries to minimize it by choosing an optimal jamming strategy $y^* \in \mathcal{Y}$. The capacity $C(x^*, y^*)$ in (5) can be achieved when a saddle point strategy pair (x^*, y^*) is chosen, which is characterized by [2], [15]

$$C(x, y^*) \leq C(x^*, y^*) \leq C(x^*, y), \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}. \tag{6}$$

This implies that: with strategy x^*, the minimal capacity that can be achieved by the authorized user is $C(x^*, y^*)$, no matter which strategy is applied by the jammer; on the other hand, if the jammer applies strategy y^*, the maximal capacity that can be achieved by the authorized user is also $C(x^*, y^*)$, no matter which strategy is applied by the authorized user. As a result, to find the optimal transmission strategy and the worst jamming strategy under the power constraints P_s and P_J, we need to find the saddle point strategy pair (x^*, y^*).

III. Optimal Strategy for Multiband Communications under Jamming over AWGN Channels

Recall that K_s denotes the number of subchannels activated by the authorized user, and K_J the number of subchannels interfered by the jammer. In this section, we derive the saddle point strategy pair (x^*, y^*) in two steps: (1) For any fixed K_s and K_J with $1 \leq K_s, K_J \leq N_c$, calculate the corresponding minimax capacity and denote it by $\widetilde{C}(K_s, K_J)$. Let $K_s = 1, 2, ..., N_c$ and $K_J = 1, 2, ..., N_c$, we can obtain an $N_c \times N_c$ payoff matrix $\widetilde{\mathbf{C}}$. (2) For the derived payoff matrix $\widetilde{\mathbf{C}}$, locate its saddle point, and then the minimax capacity of the authorized user in (5) can be calculated accordingly.

A. The Minimax Problem for Fixed K_s and K_J

With fixed K_s and K_J, the strategy space for the authorized user becomes $\widetilde{\mathcal{X}}_{K_s} = \{(K_s, \boldsymbol{\omega}_s, \mathbf{P}_s) \mid K_s \text{ Fixed}, \boldsymbol{\omega}_s \in \mathcal{W}_{s,K_s}, \mathbf{P}_s \in \mathcal{P}_{s,K_s}\} \subset \mathcal{X}$, and similarly the strategy space for the jammer becomes $\widetilde{\mathcal{Y}}_{K_J} = \{(K_J, \boldsymbol{\omega}_J, \mathbf{P}_J) \mid K_J \text{ Fixed}, \boldsymbol{\omega}_J \in \mathcal{W}_{J,K_J}, \mathbf{P}_J \in \mathcal{P}_{J,K_J}\} \subset \mathcal{Y}$. It should be noted that the user-activated subchannels and the jammed subchannels may vary from time to time, although the total number of the user-activated or jammed subchannels is fixed.

We first present two lemmas on the concavity/convexity property of two real-valued functions that will be used afterwards. More information on concavity and convexity can be found in [31].

***Lemma* 1.** *For any $v \geq 0$ and $a > 0$, the real-valued function, $f(v) = \log_2(1 + \frac{v}{a})$, is concave.*

Proof: The second-order derivative, $f''(v) = -\frac{1}{\ln 2} \frac{1}{(v+a)^2} < 0$, for any $v \geq 0$ and $a > 0$. ■

***Lemma* 2.** *For any $v \geq 0$, $a > 0$ and $b > 0$, the real-valued function, $f(v) = \log_2(1 + \frac{a}{v+b})$, is convex.*

Proof: The second-order derivative, $f''(v) = \frac{a}{\ln 2} \frac{(2v+a+2b)}{(v+a)^2 (v+a+b)^2} > 0$, for any $v \geq 0$, $a > 0$ and $b > 0$. ■

The solution[2] to the minimax problem for fixed K_s and K_J is given in Theorem 1.

***Theorem* 1.** *Let K_s be the number of subchannels activated by the authorized user, and K_J the number of subchannels interfered by the jammer. For any fixed (K_s, K_J) pair, the saddle point of $C(x, y)$ under the power constraints P_s and P_J for $x \in \widetilde{\mathcal{X}}_{K_s}$ and $y \in \widetilde{\mathcal{Y}}_{K_J}$ is reached when both authorized user and the jammer choose to apply uniform subchannel selection and uniform power allocation strategy. That is, for fixed K_s and K_J, the saddle point strategy pair $(\widetilde{x}^*, \widetilde{y}^*)$ that*

[2]The uniqueness of the solution is discussed in Appendix B.

satisfies

$$C(\widetilde{x}, \widetilde{y}^*) \leq C(\widetilde{x}^*, \widetilde{y}^*) \leq C(\widetilde{x}^*, \widetilde{y}), \quad \forall \widetilde{x} \in \widetilde{\mathcal{X}}_{K_s}, \widetilde{y} \in \widetilde{\mathcal{Y}}_{K_J}, \tag{7}$$

is given by $\widetilde{x}^* = (K_s, \boldsymbol{\omega}_s^*, \boldsymbol{P}_s^*)$ *with*

$$\begin{cases} \omega_{s,m}^* = K_s/N_c, & m = 1, 2, ..., N_c, \\ P_{s,n}^* = P_s/K_s, & n = 1, 2, ..., K_s, \end{cases} \tag{8}$$

and $\widetilde{y}^* = (K_J, \boldsymbol{\omega}_J^*, \boldsymbol{P}_J^*)$ *with*

$$\begin{cases} \omega_{J,m}^* = K_J/N_c, & m = 1, 2, ..., N_c, \\ P_{J,n}^* = P_J/K_J, & n = 1, 2, ..., K_J. \end{cases} \tag{9}$$

In this case, the minimax capacity of the authorized user is given by

$$\begin{aligned} \widetilde{C}(K_s, K_J) =& K_s \frac{K_J}{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_J/K_J + P_N/N_c}\right) \\ &+ K_s \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_N/N_c}\right). \end{aligned} \tag{10}$$

Proof: (1) We first prove that the $(\widetilde{x}^*, \widetilde{y}^*)$ pair defined in (8) and (9) satisfies the left part of (7), $C(\widetilde{x}, \widetilde{y}^*) \leq C(\widetilde{x}^*, \widetilde{y}^*)$. Assume the jammer applies the strategy $\widetilde{y}^*$, which means uniform subchannel selection and uniform power allocation as indicated in (9). For the authorized user who applies an arbitrary strategy $\widetilde{x} \in \widetilde{\mathcal{X}}_{K_s}$, we specified the indices of the activated K_s subchannels as $n = 1, 2, ..., K_s$. With any subchannel selection probability vector $\boldsymbol{\omega}_s \in \mathcal{W}_{s,K_s}$, for each subchannel activated by the authorized user, the probability that it is jammed is always $\frac{K_J}{N_c}$, since the jammer jams each subchannel with a uniform probability $\omega_{J,m}^* = \frac{K_J}{N_c}$, for any $m = 1, 2, ..., N_c$. Accordingly, the probability that each subchannel is not jammed is $1 - \frac{K_J}{N_c}$.

Considering all the subchannels activated by the authorized user, when the authorized user applies an arbitrary strategy $\widetilde{x} \in \widetilde{\mathcal{X}}_{K_s}$, and the jammer applies strategy $\widetilde{y}^*$, the ergodic capacity can be calculated as the weighted average of the capacity under jamming and the capacity in the jamming-free case,

$$\begin{aligned} C(\widetilde{x}, \widetilde{y}^*) =& \sum_{n=1}^{K_s} \left[\frac{K_J}{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_{s,n}}{P_J/K_J + P_N/N_c}\right) \right. \\ &\left. + \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_{s,n}}{P_N/N_c}\right) \right] \\ =& \frac{K_J}{N_c} \frac{B}{N_c} \sum_{n=1}^{K_s} \log_2 \left(1 + \frac{P_{s,n}}{P_J/K_J + P_N/N_c}\right) \\ &+ \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \sum_{n=1}^{K_s} \log_2 \left(1 + \frac{P_{s,n}}{P_N/N_c}\right). \end{aligned} \tag{11}$$

Note that $\sum_{n=1}^{K_s} P_{s,n} = P_s$, and applying the concavity property proved in Lemma 1, we have

$$\begin{aligned} C(\widetilde{x}, \widetilde{y}^*) \leq& K_s \frac{K_J}{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_J/K_J + P_N/N_c}\right) \\ &+ K_s \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_N/N_c}\right) \\ =& C(\widetilde{x}^*, \widetilde{y}^*), \end{aligned} \tag{12}$$

where the equality holds if and only if $P_{s,n} = \frac{P_s}{K_s}, \forall n$.

(2) Proof of the right part of (7), $C(\widetilde{x}^*, \widetilde{y}^*) \leq C(\widetilde{x}^*, \widetilde{y})$. In this part of the proof, we will show that applying uniform subchannel selection and uniform power allocation strategy $\widetilde{x}^*$ at the authorized user side guarantees a lower bound on its capacity, no matter what strategy is applied by the jammer. Assume the authorized user applies the strategy $\widetilde{x}^*$ as indicated in (8). For the jammer who applies an arbitrary strategy $\widetilde{y} \in \widetilde{\mathcal{Y}}_{K_J}$, we specified the indices of the jammed K_J subchannels as $n = 1, 2, ..., K_J$. With any subchannel selection probability vector $\boldsymbol{\omega}_J \in \mathcal{W}_{J,K_J}$, for each jammed or jamming-free subchannel, the probability that it serves as a subchannel activated by the authorized user is always $\frac{K_s}{N_c}$. Hence, the average number[3] of jammed subchannels which are also activated by the authorized user is $\frac{K_J K_s}{N_c}$, and the average number of jamming-free subchannels which are activated by the authorized user would be $(N_c - K_J)\frac{K_s}{N_c} = K_s(1 - \frac{K_J}{N_c})$.

Considering both the jammed and jamming-free subchannels, when the jammer applies an arbitrary strategy $\widetilde{y} \in \widetilde{\mathcal{Y}}_{K_J}$, and the authorized user applies strategy $\widetilde{x}^*$, the ergodic capacity can be calculated as

$$\begin{aligned} C(\widetilde{x}^*, \widetilde{y}) =& \sum_{n=1}^{K_J} \frac{K_s}{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_{J,n} + P_N/N_c}\right) \\ &+ K_s \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_N/N_c}\right). \end{aligned} \tag{13}$$

Note that $\sum_{n=1}^{K_J} P_{J,n} = P_J$, and applying the convexity property proved in Lemma 2, we have

$$\begin{aligned} C(\widetilde{x}^*, \widetilde{y}) \geq& K_s \frac{K_J}{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_J/K_J + P_N/N_c}\right) \\ &+ K_s \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_N/N_c}\right) \\ =& C(\widetilde{x}^*, \widetilde{y}^*), \end{aligned} \tag{14}$$

where the equality holds if and only if $P_{J,n} = \frac{P_J}{K_J}, \forall n$. ■

B. Capacity Optimization over K_s and K_J

In Section III-A, we derived the closed-form minimax capacity of the authorized user for fixed K_s and K_J. Considering all possible K_s and K_J, we would have an $N_c \times N_c$ matrix $\widetilde{\mathbf{C}}$, in which $\widetilde{C}(K_s, K_J)$ is the minimax capacity of the authorized user for fixed K_s and K_J, as indicated in (10). Now finding the minimax capacity in (5) can be reduced to finding the saddle point of the matrix $\widetilde{\mathbf{C}}$, that is, the entry $\widetilde{C}(i, j)$, which is simultaneously the minimum of the ith row and the maximum of the jth column.

To locate the saddle point of matrix $\widetilde{\mathbf{C}}$, we need Lemma 3.

***Lemma* 3.** *For the capacity function*

$$\begin{aligned} \widetilde{C}(K_s, K_J) =& K_s \frac{K_J}{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_J/K_J + P_N/N_c}\right) \\ &+ K_s \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_N/N_c}\right), \end{aligned} \tag{15}$$

[3] The ensemble average might not be an integer. Nevertheless, the capacity calculation would still be accurate from a statistical perspective.

we have

$$\frac{\partial \widetilde{C}}{\partial K_s} > 0, \textit{ for any } K_s = 1, 2, ..., N_c, \tag{16}$$

and

$$\frac{\partial \widetilde{C}}{\partial K_J} < 0, \textit{ for any } K_J = 1, 2, ..., N_c. \tag{17}$$

Proof: See Appendix C. ∎

Following Lemma 3, we have the following theorem.

***Theorem* 2.** *The saddle point of matrix $\widetilde{\boldsymbol{C}}$ is indexed by $(K_s^*, K_J^*) = (N_c, N_c)$. Equivalently, for all $1 \leq K_s, K_J \leq N_c$, we have*

$$\widetilde{C}(K_s, N_c) \leq \widetilde{C}(N_c, N_c) \leq \widetilde{C}(N_c, K_J). \tag{18}$$

Combining Theorems 1 and 2, we can obtain the saddle point to the original minimax problem in (5) over strategy spaces $\mathcal{X}$ and $\mathcal{Y}$. The result is summarized in Theorem 3.

***Theorem* 3.** *Assume that an authorized user and a jammer are operating against each other over the same AWGN channel consisting of N_c subchannels. Either for the authorized user to maximize its capacity, or for the jammer to minimize the capacity of the authorized user, the best strategy for both of them is to distribute the signal power or jamming power uniformly over all the N_c subchannels. In this case, the minimax capacity of the authorized user is given by*

$$C = B \log_2 \left(1 + \frac{P_s}{P_J + P_N}\right), \tag{19}$$

where B is the bandwidth of the overall spectrum, P_N the noise power, P_s and P_J the total power for the authorized user and the jammer, respectively.

Proof: The proof follows directly from Theorems 1 and 2. The minimax capacity in (19) can be derived simply by substituting $K_s = K_J = N_c$ into (10). ∎

IV. Optimal Strategy for Multiband Communications under Jamming over Frequency Selective Fading Channels

In this section, we investigate the optimal strategies for both the authorized user and the jammer in multiband communications under frequency selective fading channels.

A. The Minimax Problem for Fading Channels

Recall that the power allocation for the authorized user is characterized using the vector $\mathbf{P}_s = [P_{s,1}, P_{s,2}, ..., P_{s,N_c}]$, where $P_{s,i}$ denotes the power allocated to the ith subchannel, and $\sum_{i=1}^{N_c} P_{s,i} = P_s$ is the signal power constraint. Similarly, the power allocation vector for the jammer is $\mathbf{P}_J = [P_{J,1}, P_{J,2}, ..., P_{J,N_c}]$, and $\sum_{i=1}^{N_c} P_{J,i} = P_J$ is the jamming power constraint. As in the OFDM systems, here we assume that all the subchannels are narrowband and have flat magnitude spectrum. Let $\mathbf{H}_s = [H_{s,1}, H_{s,2}, ..., H_{s,N_c}]$ be the frequency domain channel response vector for the authorized user, and $\mathbf{H}_J = [H_{J,1}, H_{J,2}, ..., H_{J,N_c}]$ the frequency domain channel response vector for the jammer, respectively. Under the settings specified above, the capacity of the authorized user can be calculated as

$$\begin{aligned} C(\mathbf{P}_s, \mathbf{P}_J) &= \sum_{i=1}^{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{|H_{s,i}|^2 P_{s,i}}{|H_{J,i}|^2 P_{J,i} + \sigma_n^2}\right) \\ &= \sum_{i=1}^{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_{s,i}}{\frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i} + \sigma_{n,i}^2}\right), \end{aligned} \tag{20}$$

where $\sigma_n^2 = \frac{P_N}{N_c}$ is the original noise power for each subchannel, and $\sigma_{n,i}^2 = \frac{\sigma_n^2}{|H_{s,i}|^2}$.

Define $\mathcal{P}_s = \{\mathbf{P}_s = [P_{s,1}, P_{s,2}, ..., P_{s,N_c}] \mid 0 \leq P_{s,i} \leq P_s, \sum_{i=1}^{N_c} P_{s,i} = P_s\}$, and $\mathcal{P}_J = \{\mathbf{P}_J = [P_{J,1}, P_{J,2}, ..., P_{J,N_c}] \mid 0 \leq P_{J,i} \leq P_J, \sum_{i=1}^{N_c} P_{J,i} = P_J\}$. The minimax capacity of the authorized user is defined as

$$C(\mathbf{P}_s^*, \mathbf{P}_J^*) = \max_{\mathbf{P}_s^* \in \mathcal{P}_s} \min_{\mathbf{P}_J^* \in \mathcal{P}_J} C(\mathbf{P}_s, \mathbf{P}_J) = \min_{\mathbf{P}_J^* \in \mathcal{P}_J} \max_{\mathbf{P}_s^* \in \mathcal{P}_s} C(\mathbf{P}_s, \mathbf{P}_J). \tag{21}$$

As before, the authorized user tries to apply optimal signal power allocation $\mathbf{P}_s^* \in \mathcal{P}_s$ to maximize its capacity, while the jammer tries to minimize it by applying optimal jamming power allocation $\mathbf{P}_J^* \in \mathcal{P}_J$.

***Theorem* 4.** *Assume that there are N_c available subchannels. Let $\boldsymbol{H}_s = [H_{s,1}, H_{s,2}, ..., H_{s,N_c}]$ and $\boldsymbol{H}_J = [H_{J,1}, H_{J,2}, ..., H_{J,N_c}]$ denote the frequency domain channel response vector for the authorized user and the jammer, respectively. Assuming zero-mean white Gaussian noise of variance σ_n^2 over the entire band, let $\boldsymbol{\sigma}_n^2 = [\sigma_{n,1}^2, \sigma_{n,2}^2, ..., \sigma_{n,N_c}^2]$, where $\sigma_{n,i}^2 = \frac{\sigma_n^2}{|H_{s,i}|^2}$. The optimal power-allocation pair for the authorized user and the jammer under the signal power constraint $\sum_{i=1}^{N_c} P_{s,i}^* = P_s$ and the jamming power constraint $\sum_{i=1}^{N_c} P_{J,i}^* = P_J$, $(\boldsymbol{P}_s^*, \boldsymbol{P}_J^*)$, which satisfies*

$$C(\boldsymbol{P}_s, \boldsymbol{P}_J^*) \leq C(\boldsymbol{P}_s^*, \boldsymbol{P}_J^*) \leq C(\boldsymbol{P}_s^*, \boldsymbol{P}_J), \ \forall \boldsymbol{P}_s \in \mathcal{P}_s, \boldsymbol{P}_J \in \mathcal{P}_J, \tag{22}$$

can be characterized by

$$\begin{cases} P_{J,i}^* = sgn(P_{s,i}^*) \left(c_1 - \frac{|H_{s,i}|^2}{|H_{J,i}|^2} \sigma_{n,i}^2\right)^+, \ \forall i, & \text{(23a)} \\ P_{s,i}^* = \left(c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i}^* - \sigma_{n,i}^2\right)^+, \quad \forall i, & \text{(23b)} \end{cases}$$

where $(x)^+ = \max\{0, x\}$, $sgn(\cdot)$ is the sign function, and c_1, c_2 are constants determined by the power constraints.

Proof: (1) We first prove that the $(\mathbf{P}_s^*, \mathbf{P}_J^*)$ pair defined in (23) satisfies the left part of (22), $C(\mathbf{P}_s, \mathbf{P}_J^*) \leq C(\mathbf{P}_s^*, \mathbf{P}_J^*)$, $\forall \mathbf{P}_s \in \mathcal{P}_s$. With the jammer applying power allocation $\mathbf{P}_J^*$, the equivalent jamming power for the ith subchannel after fading and equalization would be $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i}^*$, as shown in (20). Taking both the jamming and the noise into account, the overall interference and noise power level for the ith subchannel at the receiver would be $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i}^* + \sigma_{n,i}^2$. As a result, the problem now turns to be the capacity maximization problem for multiband communications with nonuniform noise power levels. To this end, it is well known that the classical water pouring algorithm produces the best solution [32]. In this particular case, the water pouring solution for optimal signal power allocation would be

$$P_{s,i}^* = \left(c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i}^* - \sigma_{n,i}^2\right)^+, \quad i = 1, 2, ..., N_c, \quad (24)$$

which maximizes the capacity of the authorized user, $C(\mathbf{P}_s^*, \mathbf{P}_J^*)$, while the jammer applying power allocation $\mathbf{P}_J^*$. Note that c_2 is a constant that should be chosen such that the power constraint for the authorized user is satisfied, i.e., $\sum_{i=1}^{N_c} P_{s,i}^* = P_s$.

(2) Proof of the right part, $C(\mathbf{P}_s^*, \mathbf{P}_J^*) \le C(\mathbf{P}_s^*, \mathbf{P}_J), \ \forall \mathbf{P}_J \in \mathcal{P}_J$. To this end, we need to find the optimal jamming power allocation $\mathbf{P}_J^*$, which can minimize the capacity of the authorized user applying power allocation $\mathbf{P}_s^*$. Let $\gamma_i = \frac{|H_{J,i}|^2}{|H_{s,i}|^2}, \ \forall i$. With the authorized user applying power allocation $\mathbf{P}_s^*$, following (20), the optimization problem for jamming power allocation can be formulated as

$$\min_{\mathbf{P}_J \in \mathcal{P}_J} \sum_{i=1}^{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_{s,i}^*}{\gamma_i P_{J,i} + \sigma_{n,i}^2}\right); \quad (25a)$$

$$s.t. \ \sum_{i=1}^{N_c} P_{J,i} = P_J, \quad (25b)$$

$$P_{J,i} \ge 0, \ \forall i. \quad (25c)$$

Note that in this optimization problem, we have both equality and inequality constraints. Hence, we need to take the Karush-Kuhn-Tucker (KKT) approach [31], which generalizes the conventional method of Lagrange multipliers by allowing inequality constraints. As observed in (24), for $P_{s,i}^* > 0$, $P_{s,i}^* = c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i}^* - \sigma_{n,i}^2$. In addition, the capacity of any subchannel with zero signal power (i.e., $P_{s,i}^* = 0$) is zero. Define

$$\begin{aligned} J(\mathbf{P}_J, \mathbf{u}, v) =& \sum_{i=1}^{N_c} \frac{B}{N_c} \log_2 \left(1 + \frac{P_{s,i}^*}{\gamma_i P_{J,i} + \sigma_{n,i}^2}\right) \\ &- u_i P_{J,i} + v \left(\sum_{i=1}^{N_c} P_{J,i} - P_J\right) \\ =& \sum_{i \in \{i | P_{s,i}^* > 0\}} \frac{B}{N_c} \log_2 \frac{c_2}{\gamma_i P_{J,i} + \sigma_{n,i}^2} \\ &- u_i P_{J,i} + v \left(\sum_{i=1}^{N_c} P_{J,i} - P_J\right), \end{aligned} \quad (26)$$

where $\mathbf{u} = [u_1, u_2, ..., u_{N_c}]$ and v are Lagrange multipliers that should satisfy the KKT conditions as below:

$$\frac{\partial J}{\partial P_{J,i}} = 0, \ \ u_i P_{J,i} = 0, \ \ u_i \ge 0, \ \ \forall i. \quad (27)$$

The first-order partial differentiation with respect to each $P_{J,i}$ can be calculated as

$$\frac{\partial J}{\partial P_{J,i}} = \begin{cases} -\frac{B}{N_c} \frac{1}{\ln 2} \frac{\gamma_i}{\gamma_i P_{J,i} + \sigma_{n,i}^2} - u_i + v, & P_{s,i}^* > 0, \\ -u_i + v, & P_{s,i}^* = 0. \end{cases} \quad (28)$$

For each subchannel with nonzero signal power (i.e., $P_{s,i}^* > 0$), applying the KKT conditions and eliminating u_i, we have

$$\begin{cases} v - \frac{B}{N_c} \frac{1}{\ln 2} \frac{\gamma_i}{\gamma_i P_{J,i} + \sigma_{n,i}^2} \ge 0, \\ P_{J,i} \left[v - \frac{B}{N_c} \frac{1}{\ln 2} \frac{\gamma_i}{\gamma_i P_{J,i} + \sigma_{n,i}^2}\right] = 0. \end{cases} \quad (29)$$

Solving (29), the optimal jamming power for the ith subchannel (with nonzero signal power) can be obtained as

$$P_{J,i}^* = \left(\frac{B}{N_c} \frac{1}{\ln 2} \frac{1}{v} - \frac{1}{\gamma_i} \sigma_{n,i}^2\right)^+. \quad (30)$$

Similarly, for each subchannel with zero signal power (i.e., $P_{s,i}^* = 0$), applying the KKT conditions and eliminating u_i, we have $v P_{J,i} = 0$. It is observed from (29) that $v > 0$, so the optimal jamming power for the ith subchannel (with zero signal power) is $P_{J,i}^* = 0$. Let $c_1 = \frac{B}{N_c} \frac{1}{\ln 2} \frac{1}{v}$, and replace γ_i with $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}$, we can summarize the result as

$$P_{J,i}^* = \begin{cases} \left(c_1 - \frac{|H_{s,i}|^2}{|H_{J,i}|^2} \sigma_{n,i}^2\right)^+, & P_{s,i}^* > 0, \\ 0, & P_{s,i}^* = 0, \end{cases} \quad (31)$$

where c_1 should be chosen such that the power constraint for the jammer is satisfied, i.e., $\sum_{i=1}^{N_c} P_{J,i}^* = P_J$. This is exactly the optimal jamming power allocation as expressed in (23a), which minimizes the capacity of the authorized user, $C(\mathbf{P}_s^*, \mathbf{P}_J^*)$, given that the authorized user applies power allocation $\mathbf{P}_s^*$. ■

B. Correlated Fading Channels: A Two-Step Water Pouring Algorithm

Theorem 4 characterizes the dynamic relationship between the optimal signal power allocation $\mathbf{P}_s^*$ and the optimal jamming power allocation $\mathbf{P}_J^*$. As shown in (23), due to the mutual dependency between $\mathbf{P}_s^*$ and $\mathbf{P}_J^*$, it is generally difficult to find an exact solution for them. However, in this subsection, we will show that if the channels of the authorized user and the jammer are correlated, the saddle point, $(\mathbf{P}_s^*, \mathbf{P}_J^*)$, can be calculated explicitly using a two-step water pouring algorithm.

***Theorem* 5.** *(A Two-Step Water Pouring Algorithm) With the same conditions as in Theorem 4, the saddle point, which indicates the optimal signal power allocation and the optimal jamming power allocation, is given by*

$$\begin{cases} P_{J,i}^* = \left(c_1 - \frac{|H_{s,i}|^2}{|H_{J,i}|^2} \sigma_{n,i}^2\right)^+, & \forall i, \quad (32a) \\ P_{s,i}^* = \left(c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i}^* - \sigma_{n,i}^2\right)^+, & \forall i, \quad (32b) \end{cases}$$

as long as

$$|H_{J,i}|^2 \le \frac{\sigma_n^2}{c_1} \ \ or \ \ \frac{|H_{J,i}|^2}{|H_{s,i}|^2} < \frac{c_2}{c_1}, \ \ \forall i, \quad (33)$$

where $(x)^+ = \max\{0, x\}$, and c_1, c_2 are constants that should be chosen such that the power constraints are satisfied, i.e., $\sum_{i=1}^{N_c} P_{s,i}^* = P_s$ and $\sum_{i=1}^{N_c} P_{J,i}^* = P_J$.

Proof: The basic idea here is that given zero signal power for a particular subchannel, it is apparently not necessary to allocate positive jamming power in that subchannel; at the same time, over all the subchannels with nonzero signal power, the optimal jamming power allocation can be formed using the water pouring algorithm. We start by applying the water pouring algorithm over all subchannels,

$$P_{J,i}^* = \left(c_1 - \frac{|H_{s,i}|^2}{|H_{J,i}|^2} \sigma_{n,i}^2 \right)^+, \;\; i = 1, 2, ..., N_c. \tag{34}$$

For the optimality of (34), we further need to ensure that $P_{J,i}^* = 0$, whenever $P_{s,i}^* = 0$.

As can be seen, a violation occurs ($P_{J,i}^* > 0$ and $P_{s,i}^* = 0$), if and only if for some subchannel indexed by i,

$$\begin{cases} P_{J,i}^* = c_1 - \frac{|H_{s,i}|^2}{|H_{J,i}|^2} \sigma_{n,i}^2 > 0, \\ c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2} P_{J,i}^* - \sigma_{n,i}^2 \leq 0, \end{cases} \tag{35}$$

which yields

$$|H_{J,i}|^2 > \frac{\sigma_n^2}{c_1} \text{ and } \frac{|H_{J,i}|^2}{|H_{s,i}|^2} \geq \frac{c_2}{c_1}. \tag{36}$$

Note that $\sigma_{n,i}^2 = \frac{\sigma_n^2}{|H_{s,i}|^2}$. Hence, the conditions characterized in (33) ensure that no violation occurs, and therefore the saddle point calculated by (32) is valid for both capacity maximization by the authorized user and capacity minimization by the jammer. ■

In the following, we consider a special case where the channels corresponding to the authorized user and the jammer are *relatively flat* with respect to each other, that is, their magnitude spectrum is proportional to each other, i.e., $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} = \gamma, \ \forall i$. As will be shown in Corollary 1, when the user channel and the jammer channel are relatively flat with respect to each other, the conditions in (33) are always satisfied, and the saddle point calculation can be simplified accordingly.

***Corollary* 1.** *With the same conditions as in Theorem 4, if the magnitude spectrum of the channels for the authorized user and the jammer is proportional to each other, i.e.,* $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} = \gamma, \ \forall i$, *the saddle point, which indicates the optimal signal power allocation and the optimal jamming power allocation, can be calculated as*

$$\begin{cases} P_{J,i}^* = \left(c_1 - \frac{1}{\gamma} \sigma_{n,i}^2 \right)^+, & \forall i, \\ P_{s,i}^* = \left(c_2 - \gamma P_{J,i}^* - \sigma_{n,i}^2 \right)^+, & \forall i, \end{cases} \tag{37}$$

where $(x)^+ = \max\{0, x\}$, *and* c_1, c_2 *are constants that should be chosen such that the power constraints are satisfied, i.e.,* $\sum_{i=1}^{N_c} P_{s,i}^* = P_s$ *and* $\sum_{i=1}^{N_c} P_{J,i}^* = P_J$.

Proof: Note that with $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} = \gamma, \ \forall i$, (32) reduces to (37). Following Theorem 5, we only need to show that the conditions specified in (33) are satisfied.

First, we show that the constants $c_1, \ c_2$ resulted from (37) and the power constraints always satisfy $\frac{c_2}{c_1} > \gamma$. This is proved by contradiction as follows. Suppose $\frac{c_2}{c_1} \leq \gamma$. Following (37), for any $i = 1, 2, ..., N_c$, $P_{J,i}^* \geq c_1 - \frac{1}{\gamma} \sigma_{n,i}^2$. Thus, $c_2 - \gamma P_{J,i}^* - \sigma_{n,i}^2 \leq c_2 - \gamma (c_1 - \frac{1}{\gamma} \sigma_{n,i}^2) - \sigma_{n,i}^2 = c_2 - \gamma c_1 \leq 0$. This implies that for all subchannels, we always have $P_{s,i}^* = \left(c_2 - \gamma P_{J,i}^* - \sigma_{n,i}^2 \right)^+ = 0$, which contradicts with the power constraint that $\sum_{i=1}^{N_c} P_{s,i}^* = P_s$. As a result, we must have $\frac{c_2}{c_1} > \gamma$.

It then follows that for any subchannel, we always have $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} = \gamma < \frac{c_2}{c_1}$. This ensures that the conditions specified in (33) are always satisfied. Hence, the solution calculated by (37) must be a valid saddle point. ■

Furthermore, if the magnitude spectrum of channels for the authorized user and the jammer is equal to each other, i.e., $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} = \gamma = 1, \ \forall i$, the two-step water pouring algorithm in (37) can be graphically illustrated in Fig. 1, where the saddle point can simply be obtained by pouring all the signal power after pouring all the jamming power into a tank with given noise power levels. We would like to point out that under AWGN channels, the noise power levels are flat; hence, the water pouring process here would result in uniform power allocation for both the jammer and the authorized user, which echoes the results in Section III.

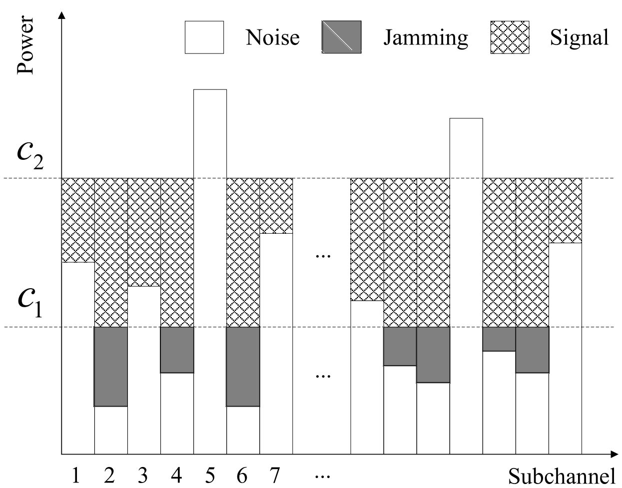


Fig. 1. Water pouring under jamming with equal channel magnitude spectrum for the authorized user and the jammer (i.e., $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} = \gamma = 1, \ \forall i$).

Discussions: Theorem 5 provides an efficient two-step water pouring algorithm to calculate the saddle point of the minimax problem. This algorithm guarantees a valid saddle point under certain conditions as illustrated in (33). Corollary 1 further shows a sufficient (but may not be necessary) condition for (33) being satisfied: the channels for the authorized user and the jammer are relatively flat with respect to each other, i.e., their magnitude spectrum is proportional to each other. From the arbitrarily varying channel (AVC) [5], [6] point of view, the correlation between the user channel and the jamming channel can be regarded as an indicator of possible symmetricity between the user and the jammer. In the case that the user channel and the jammer channel are not relatively flat with respect to each other, as shown in Section V-B, as long as the cross correlation between the two channels is reasonably high, we found that the algorithm in Theorem 5 can still provide a much better solution than uniform power allocation.

C. Arbitrary Fading Channels: An Iterative Water Pouring Algorithm

The two-step water pouring algorithm in Theorem 5 is a very efficient solution for correlated fading channels. However, if the channels of the authorized user and the jammer are not correlated, the algorithm needs to be extended. Motivated by [33], in this subsection, we will propose an iterative water pouring algorithm, which is able to find a numerical solution to the saddle point for arbitrary fading channels.

We first begin with the two-step water pouring algorithm in Theorem 5, since it is a good starting point with possibly only a few violations against (33). We can then try to remove or at least alleviate the violations identified. Recall that in the two-step water pouring algorithm, we first allocate the jamming power by

$$P_{J,i}^* = \left(c_1 - \frac{|H_{s,i}|^2}{|H_{J,i}|^2}\sigma_{n,i}^2\right)^+, \ \forall i, \tag{38}$$

which is equivalent to

$$\frac{|H_{J,i}|^2}{|H_{s,i}|^2}P_{J,i}^* = \left(\frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1 - \sigma_{n,i}^2\right)^+, \ \forall i. \tag{39}$$

The physical meaning of (39) is that: for each subchannel i with positive jamming power allocation (i.e., $P_{J,i}^* > 0$), the effective jamming power, $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}P_{J,i}^*$, plus the noise power level, $\sigma_{n,i}^2$, should be $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1$. However, if $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1 > c_2$, the allocated jamming power for this subchannel would be *more than necessary*. The underlying argument is that: to prevent or discourage the transmission of the authorized user in a particular subchannel, it would be good enough to make sure the water level after jamming power allocation, $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}P_{J,i}^* + \sigma_{n,i}^2$, reaches c_2. In this case, according to (32b), the authorized user would have already been discouraged from allocating any power in this subchannel. Hence, any jamming power that results into a water level higher than c_2 would be more than necessary. Based on the reasoning above, (39) should be revised to

$$\frac{|H_{J,i}|^2}{|H_{s,i}|^2}P_{J,i}^* = \left[\min\left(\frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1, c_2\right) - \sigma_{n,i}^2\right]^+, \ \forall i, \tag{40}$$

which is equivalent to

$$P_{J,i}^* = \left[\min\left(c_1, \frac{|H_{s,i}|^2}{|H_{J,i}|^2}c_2\right) - \frac{|H_{s,i}|^2}{|H_{J,i}|^2}\sigma_{n,i}^2\right]^+, \ \forall i. \tag{41}$$

Replacing the jamming power allocation in Theorem 5 by (41), we have

$$\begin{cases} P_{J,i}^* = \left[\min\left(c_1, \frac{|H_{s,i}|^2}{|H_{J,i}|^2}c_2\right) - \frac{|H_{s,i}|^2}{|H_{J,i}|^2}\sigma_{n,i}^2\right]^+, \ \forall i, & \text{(42a)} \\ P_{s,i}^* = \left(c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2}P_{J,i}^* - \sigma_{n,i}^2\right)^+, \quad \forall i. & \text{(42b)} \end{cases}$$

Then we can approximate the optimal power allocation pair by alternatively running (42a) and (42b) until it converges. Following this idea, we propose an iterative water pouring algorithm, which is summarized in Table I.

TABLE I
THE ITERATIVE WATER POURING ALGORITHM.

Step 1. Run the two-step water pouring algorithm once:

1) Allocate jamming power by
$P_{J,i}^* = \left(c_1 - \frac{|H_{s,i}|^2}{|H_{J,i}|^2}\sigma_{n,i}^2\right)^+, \ \forall i;$
2) Allocate user signal power by
$P_{s,i}^* = \left(c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2}P_{J,i}^* - \sigma_{n,i}^2\right)^+, \ \forall i.$

Step 2. Exit if no violations, i.e., $|H_{J,i}|^2 \le \frac{\sigma_n^2}{c_1}$ *or* $\frac{|H_{J,i}|^2}{|H_{s,i}|^2} < \frac{c_2}{c_1}$, $\forall i$;

Step 3. Repeat the following water pouring steps until convergence:

1) Allocate jamming power by
$P_{J,i}^* = \left[\min\left(c_1, \frac{|H_{s,i}|^2}{|H_{J,i}|^2}c_2\right) - \frac{|H_{s,i}|^2}{|H_{J,i}|^2}\sigma_{n,i}^2\right]^+, \ \forall i;$
2) Allocate user signal power by
$P_{s,i}^* = \left(c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2}P_{J,i}^* - \sigma_{n,i}^2\right)^+, \ \forall i.$

It should be noted that: for the jamming power allocation in step 3, c_2 is known from previous calculations. As a result, for each water pouring step throughout the algorithm, there is strictly only one constant unknown, and it can be determined by an efficient binary search algorithm [34]. The convergence analysis of the iterative water pouring algorithm can be found in Appendix D.

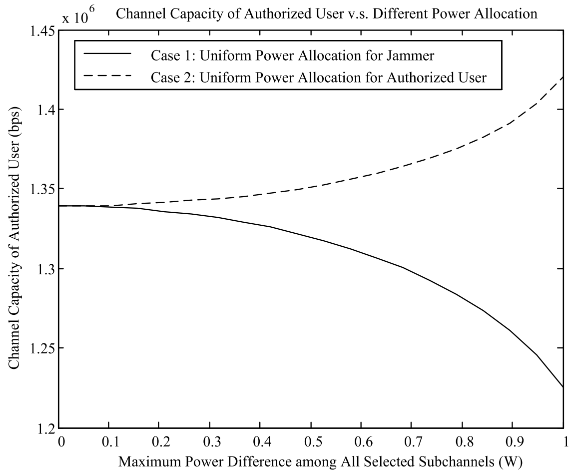
V. NUMERICAL RESULTS

In this section, we evaluate the impact of different strategies applied by the authorized user and the jammer on the capacity of the authorized user through numerical examples. In the following, we assume $N_c = 64$, $B = 1$ MHz, $P_s = P_J = 16$ W. Both AWGN channels and frequency selective fading channels are evaluated.

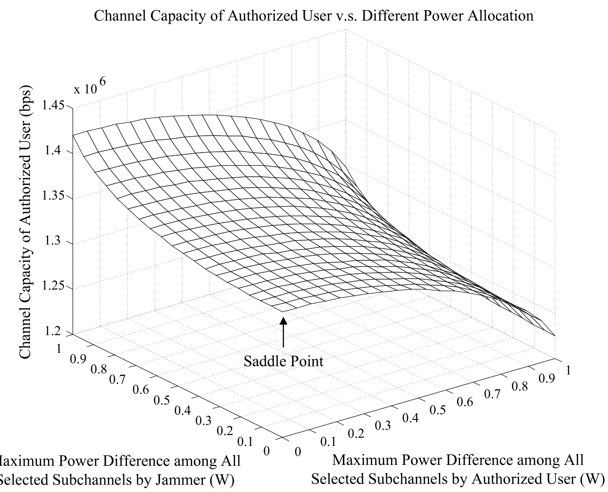
A. AWGN Channels

In this subsection, we investigate AWGN channels, where the overall signal-to-noise ratio (SNR) is set to 10dB. In light of Theorem 1, we assume that both the authorized user and the jammer apply uniform subchannel selection, that is, all subchannels are equally probable to be selected.

1) Capacity v.s. Power Allocation with Fixed K_s and K_J In this example, we evaluate the capacity of the authorized user under different transmit and jamming power allocation schemes. We set the power allocation vector as one whose elements, if sorted, would form an arithmetic sequence, and we use the *maximum power difference* among all the selected subchannels as the metric of uniformity. Hence, the maximum power difference indicates how far the power allocation is away from being uniform, and a zero difference means uniform power allocation. Fig. 2 shows the results when both the authorized user and the jammer select half of all the available subchannels each time, while Fig. 3 corresponds to the case where both of them select all the available subchannels. In the 2D view, we evaluate the capacity in two scenarios: (1) uniform jamming power allocation, while the power allocation for the authorized user is nonuniform; (2) the case which is exactly opposite to (1). The 3D counterpart in these two figures provides spacial views on the physical meanings of the derived saddle points. Note that the saddle point is reached at one of the vertices, hence the 3D view includes only a quarter portion of a regular saddle-point graph.



(a) 2D view.

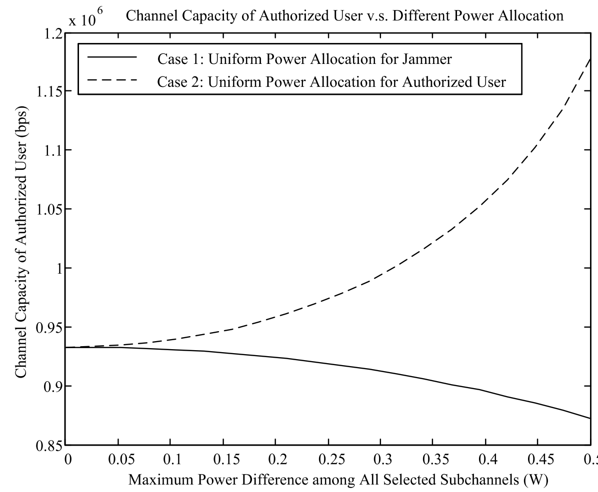


(b) 3D view.

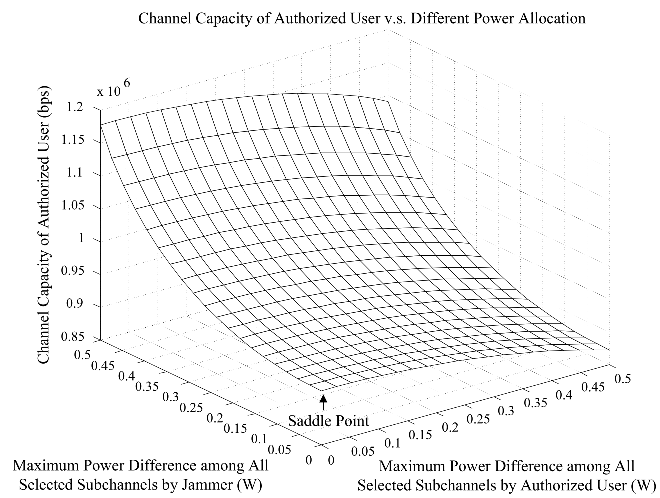
Fig. 2. AWGN channels: channel capacity of given bandwidth (1 MHz) v.s. different power allocation. Both the authorized user and the jammer select half of all the available subchannels each time.

From Fig. 2 and Fig. 3, *it can be seen that*, when the number of user-activated subchannels K_s and the number of jammed subchannels K_J are both fixed: (1) if the jammer applies uniform power allocation, the authorized user maximizes its capacity when it applies uniform power allocation as well; (2) if the authorized user applies uniform power allocation, the jammer minimizes the capacity of the authorized user when it applies uniform power allocation as well; (3) the minimax capacity (the intersections in 2D view and the labeled saddle points in 3D view) serves as a lower bound when the authorized user applies uniform power allocation under all possible jamming power allocation schemes, and simultaneously it serves as an upper bound when the jammer applies uniform power allocation under all possible signal power allocation schemes. The results above match well with Theorem 1.

2) Capacity v.s. Number of Selected Subchannels In this example, we evaluate the capacity of the authorized user with



(a) 2D view.



(b) 3D view.

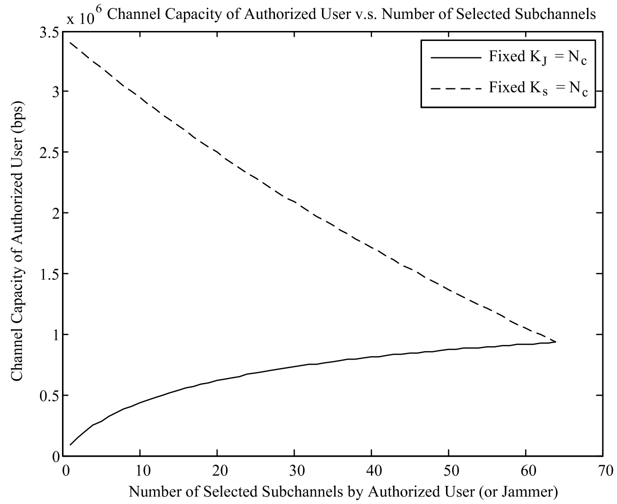
Fig. 3. AWGN channels: channel capacity of given bandwidth (1 MHz) v.s. different power allocation. Both the authorized user and the jammer always select all the available subchannels.

different number of selected subchannels by the authorized user or the jammer. For each possible pair (K_s, K_J), both the authorized user and the jammer apply uniform power allocation. *It is observed in Fig. 4 that* the best strategy is to utilize all the N_c subchannels, either for the authorized user to maximize its capacity, or for the jammer to minimize the capacity of the authorized user. This result matches well with Theorem 2.

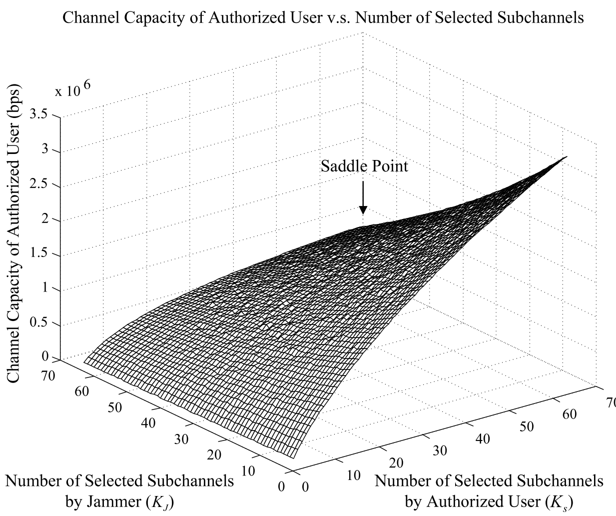
B. Frequency Selective Fading Channels

In this subsection, we investigate frequency selective fading channels. Both the two-step water pouring algorithm for correlated fading channels and the iterative water pouring algorithm for arbitrary fading channels are evaluated.

1) Two-Step Water Pouring Algorithm for Correlated Fading Channels To address the correlation between channels for the authorized user and the jammer, we introduce a correlation index, $\lambda (0 \leq \lambda \leq 1)$, which characterizes how much dependence the two channels have on each other. More specifically, in this simulation example, we generate



(a) 2D view.



(b) 3D view.

Fig. 4. AWGN channels: channel capacity of given bandwidth (1 MHz) v.s. number of selected subchannels.

the magnitude spectrum of channels in two steps: (1) create two random vectors, $\mathbf{x}_1 = [x_{1,1}, x_{1,2}, ..., x_{1,N_c}]$ and $\mathbf{x}_2 = [x_{2,1}, x_{2,2}, ..., x_{2,N_c}]$, in which all $x_{1,i}$ and $x_{2,i}$ are independent random variables with uniform distribution over (0,1); (2) generate the magnitude spectrum of the channel for the authorized user by assigning $|H_{s,i}|^2 = x_{1,i}, \ \forall i$, and that for the jammer as $|H_{J,i}|^2 = \lambda |H_{s,i}|^2 + (1-\lambda) x_{2,i}, \ \forall i$. Particularly, $\lambda = 1$ generates equal channel magnitude spectrum for the authorized user and the jammer, while $\lambda = 0$ generates completely independent channel magnitude spectrum.

In Fig. 5, with the SNR being set to 10dB, we compare the capacity of the authorized user in three cases with different power allocation strategies: (1) both the authorized user and the jammer perform power allocation by the two-step water pouring algorithm; (2) the authorized user performs power allocation by the two-step water pouring algorithm, while the jammer performs uniform power allocation; (3) the jammer performs power allocation by the two-step water pouring algorithm, while the authorized user performs uniform power allocation.

There are *four main observations*: (1) the authorized user always has a higher capacity if he performs signal power allocation by the two-step water pouring algorithm, compared to uniform signal power allocation; (2) the capacity of the authorized user decreases significantly if the channel of the jammer is more correlated with that of the authorized user, which implies that the jammer can enhance its jamming effect by delivering jamming power through a channel that is correlated with the authorized user's channel; (3) in a more serious case with high channel correlation, the jammer can limit the capacity of the authorized user more effectively by performing jamming power allocation by the two-step water pouring algorithm, compared to uniform jamming power allocation; (4) if the jammer is not able to achieve high channel correlation, uniform jamming power allocation is preferred instead of applying the two-step water pouring algorithm.

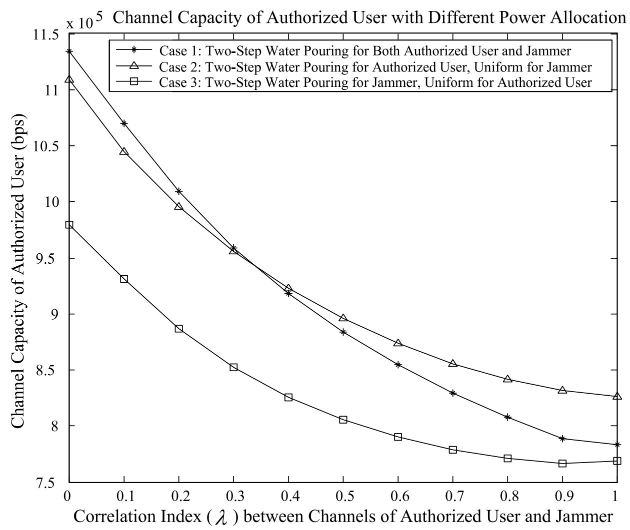


Fig. 5. Evaluation of the two-step water pouring algorithm under frequency selective fading channels: channel capacity of given bandwidth (1 MHz) with different power allocation v.s. varying channel correlation index λ.

In Fig. 6, with the channel correlation index being set to $\lambda = 0.75$, we compare the capacity of the authorized user with different power allocation versus varying SNR. *It is observed that*: (1) with reasonably high correlation between the user channel and the jamming channel, the power allocation strategy given by the two-step water pouring algorithm has a notable advantage over uniform power allocation, either for the authorized user to maximize its capacity, or for the jammer to minimize the capacity of the authorized user; (2) when the SNR is sufficiently high, the jamming power allocation produced by the two-step water pouring algorithm converges to uniform.

2) Iterative Water Pouring Algorithm for Arbitrary Fading Channels In this simulation, the channel magnitude spectrum of the authorized user and the jammer are completely independent, which is equivalent to $\lambda = 0$ in the correlated fading channel setting. Similarly, we compare the capacity of the authorized user in three cases using the same setting as in the previous example, except that the two-step water pouring algorithm is replaced by the iterative water pouring algorithm. In Fig. 7, again, *it is observed that*: the iterative

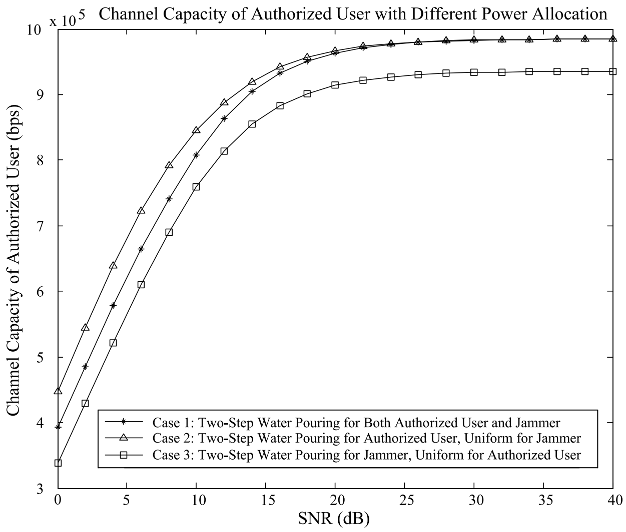


Fig. 6. Evaluation of the two-step water pouring algorithm under frequency selective fading channels: channel capacity of given bandwidth (1 MHz) with different power allocation v.s. varying SNR.

water pouring algorithm has a notable advantage over uniform power allocation, either for the authorized user to maximize its capacity, or for the jammer to minimize the capacity of the authorized user.

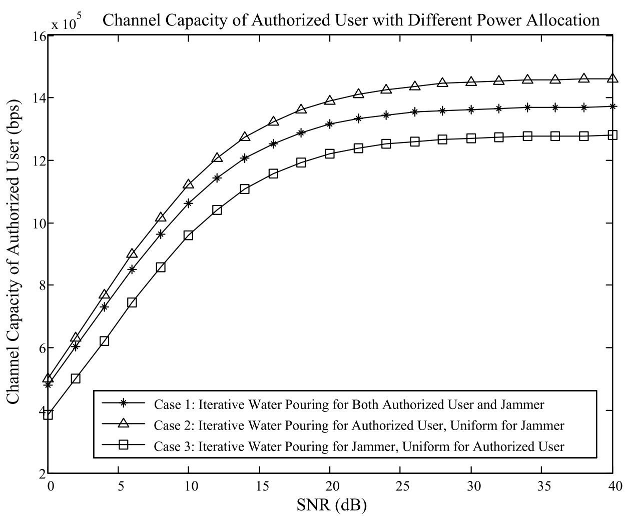


Fig. 7. Evaluation of the iterative water pouring algorithm under frequency selective fading channels: channel capacity of given bandwidth (1 MHz) with different power allocation v.s. varying SNR.

VI. Conclusions

In this paper, we considered jamming and jamming mitigation as a game between a power-limited jammer and a power-limited authorized user, who operate against each other over the same spectrum consisting of multiple bands. The strategic decision-making of the authorized user and the jammer was modeled as a two-party zero-sum game, where the payoff function is the capacity that can be achieved by the authorized user in presence of the jammer. *Under AWGN channels*, we found that either for the authorized user to maximize its capacity, or for the jammer to minimize the capacity of the authorized user, the best strategy for both of them is to distribute the signal power or jamming power uniformly over all the available spectrum. *Under frequency selective fading channels*, we first characterized the dynamic relationship between the optimal signal power allocation and the optimal jamming power allocation in the minimax game, and then proposed an iterative water pouring algorithm to find the optimal power allocation schemes for both the authorized user and the jammer. Numerical results were provided to demonstrate the effectiveness of the proposed strategies for both AWGN and frequency selective fading channels.

Appendix A
Subchannel Selection with Nonuniform Preferences

This appendix provides an approach to select K out of N_c subchannels according to a probability vector $\boldsymbol{\omega} = [\omega_1, \omega_2, ..., \omega_{N_c}]$, where ω_m denotes the probability that the mth subchannel is selected each time, and $\sum_{m=1}^{N_c} \omega_m = K$. Suppose ω_m's are rational numbers, then there exists a finite positive integer M, such that $l_m = M\omega_m$ is a positive integer for all $1 \le m \le N_c$. Furthermore, we have $\sum_{m=1}^{N_c} l_m = KM$. The proposed approach works with the following steps:

1) Construct a $K \times M$ matrix, in which the kth ($1 \le k \le M$) column represents the kth subchannel selection result; Prepare l_m balls labeled "subchannel m" for all $1 \le m \le N_c$, and there are $\sum_{m=1}^{N_c} l_m = KM$ balls in total;
2) Initialization: set $k = 1$ as the current row to be filled, $m = 1$ as the current subchannel to be worked on, and $r = M$ as the number of empty entries for the current row;
3) Select l_1 entries randomly from the 1st ($k = 1$) row of the matrix, and fill them with all the l_1 balls. For $k \ge 1$ and $m \ge 2$, placement of the l_m balls labeled "subchannel m" has two cases:
   - If $l_m \le r$, the current row has a capacity large enough to accommodate all the l_m balls. Select l_m entries randomly from the kth row of the matrix, and fill them with all the l_m balls. Update the number of empty entries for the current row by $r \leftarrow (r - l_m)$; if all empty entries of the current row are filled, move to the next row by setting $k \leftarrow (k+1)$ and $r \leftarrow M$.
   - If $l_m > r$, the l_m balls have to be split into the current row and the next row. First fill the r empty entries of the kth row with r out of l_m balls; then select $l_m - r$ out of $M - r$ entries randomly from the $(k+1)$th row, and fill them with the remaining $l_m - r$ balls. Note that there are only $M - r$ entries in the new row available here, since the r columns already containing balls labeled "subchannel m" have to be avoided. Update the number of empty entries for the current row by $r \leftarrow [M - (l_m - r)]$, and set the current row by $k \leftarrow (k + 1)$.
4) Set $m \leftarrow (m + 1)$ and repeat 4) until all KM balls are placed in the $K \times M$ matrix;

5) Fetch each column in the matrix to generate the subchannel selection results for M consecutive time slots, and repeat all the steps above until all information transmission is done.

In the following, we justify that the probability of the mth subchannel being selected each time is exactly the desired ω_m. For each possible $1 \le m \le N_c$, the number of balls labeled "subchannel m" is $l_m = M\omega_m \le M$. According to the approach above, all the l_m balls can be placed into at most two rows in the matrix. Denote $\mathcal{P}_{m,k}$ as the probability that the mth subchannel is chosen in the kth row. Then $\mathcal{P}_{m,k} = \frac{r_k}{M}$, where r_k is the number of balls labeled "subchannel m" that have been placed in the kth row of the matrix, since the mth subchannel would appear r_k times in the kth place out of the total M times of subchannel selection. If the l_m balls are placed into only one row, e.g., the k_0th row, for each subchannel selection, $\mathcal{P}_{m,k} = \frac{l_m}{M}$ for $k = k_0$, and zero elsewhere. Hence, the probability that the mth subchannel is selected considering all possible places would be $\mathcal{P}_m = \sum_{k=1}^{K} \mathcal{P}_{m,k} = \mathcal{P}_{m,k_0} = \frac{l_m}{M} = \omega_m$. If they are placed into two consecutive rows, e.g., the k_0th row and the (k_0+1)th row, then $\mathcal{P}_{m,k} = \frac{r}{M}$ for $k = k_0$, $\mathcal{P}_{m,k} = \frac{l_m - r}{M}$ for $k = k_0+1$, and zero elsewhere. In this case, $\mathcal{P}_m = \sum_{k=1}^{K} \mathcal{P}_{m,k} = \mathcal{P}_{m,k_0} + \mathcal{P}_{m,k_0+1} = \frac{r}{M} + \frac{l_m - r}{M} = \omega_m$. As a result, we can conclude that the probability that the mth subchannel is selected resulted from the proposed approach is $\mathcal{P}_m = \omega_m$.

APPENDIX B
UNIQUENESS OF THE SOLUTION TO THEOREM 1

It is stated in Theorem 1 that: assuming there are K_s subchannels activated by the authorized user and K_J subchannels interfered by the jammer, the saddle point is reached when both authorized user and the jammer choose to apply uniform subchannel selection and uniform power allocation strategy. In this appendix, we show the uniqueness of this solution.

First, it is impossible to have nonuniform subchannel selection in a saddle point strategy pair. The reason is that: if either the authorized user or the jammer applies nonuniform subchannel selection, the nonuniform pattern could be detected and utilized by the other. That is, the authorized user could avoid the subchannels that are highly likely interfered by the jammer, while the jammer would prefer to interfere the subchannels that are highly likely used by the authorized user.

Second, with uniform subchannel selection for both the authorized user and the jammer, it is impossible to have nonuniform power allocation in a saddle point strategy pair. We will start with the case where the authorized user tries to maximize its capacity. We assume that the jamming power allocation (not necessarily uniform) is characterized by $\mathbf{P}_J^* = [P_{J,1}^*, P_{J,2}^*, ..., P_{J,K_J}^*]$, and different jamming power levels are assigned to the jammed subchannel randomly. That is, if a subchannel is jammed, the jamming power could be any $P_{J,m}^* (1 \le m \le K_J)$ with equal probability, $\frac{1}{K_J}$. The reason for random assignment is similar to uniform subchannel selection, i.e., the signal power allocation with a fixed pattern could also be detected and utilized by the jammer. Applying a similar idea to (11) and considering all possible jamming power for each jammed subchannel, the capacity of the authorized user can be calculated as

$$\begin{aligned} C(\mathbf{P}_s, \mathbf{P}_J^*) =& \sum_{n=1}^{K_s} \left[\frac{K_J}{N_c} \sum_{m=1}^{K_J} \frac{1}{K_J} \frac{B}{N_c} \log_2 \left(1 + \frac{P_{s,n}}{P_{J,m}^* + P_N/N_c} \right) \right. \\ & \left. + \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_{s,n}}{P_N/N_c}\right) \right] \\ =& \frac{1}{N_c} \frac{B}{N_c} \sum_{m=1}^{K_J} \sum_{n=1}^{K_s} \log_2 \left(1 + \frac{P_{s,n}}{P_{J,m}^* + P_N/N_c}\right) \\ & + \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \sum_{n=1}^{K_s} \log_2 \left(1 + \frac{P_{s,n}}{P_N/N_c}\right). \end{aligned} \tag{43}$$

Note that $\sum_{n=1}^{K_s} P_{s,n} = P_s$, and applying the concavity property proved in Lemma 1, we have

$$\sum_{n=1}^{K_s} \log_2 \left(1 + \frac{P_{s,n}}{P_{J,m}^* + P_N/N_c}\right) \le K_s \log_2 \left(1 + \frac{P_s/K_s}{P_{J,m}^* + P_N/N_c}\right), \tag{44}$$

and

$$\sum_{n=1}^{K_s} \log_2 \left(1 + \frac{P_{s,n}}{P_N/N_c}\right) \le K_s \log_2 \left(1 + \frac{P_s/K_s}{P_N/N_c}\right). \tag{45}$$

Substituting (44) and (45) into (43), we have

$$\begin{aligned} C(\mathbf{P}_s, \mathbf{P}_J^*) \le& \frac{K_s}{N_c} \frac{B}{N_c} \sum_{m=1}^{K_J} \log_2 \left(1 + \frac{P_s/K_s}{P_{J,m}^* + P_N/N_c}\right) \\ & + K_s \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \log_2 \left(1 + \frac{P_s/K_s}{P_N/N_c}\right), \end{aligned} \tag{46}$$

where the equality holds if and only if $P_{s,n} = \frac{P_s}{K_s}, \forall n$. So far, we have proved that: with random jamming power allocation, the authorized user can maximize its capacity only by uniform power allocation. For the jammer to minimize the capacity of the authorized user, we can obtain a similar result by applying the same method.

APPENDIX C
PROOF OF LEMMA 3

To prove Lemma 3, we need the following result:

Lemma 4. *For a real-valued function $f(v) = \ln(1+v) - \frac{v}{1+v}$, $f(v) > 0$, for any $v > 0$.*

Proof: When $v > 0$, $f'(v) = \frac{v}{(1+v)^2} > 0$. Thus, $f(v) > f(0) = 0$. ∎

Now we are ready to prove Lemma 3.

(1) The first-order derivative of $\widetilde{C}$ over K_s,

$$\begin{aligned} \frac{\partial \widetilde{C}}{\partial K_s} =& \frac{K_J}{N_c} \frac{B}{N_c} \frac{1}{\ln 2} \left[\ln \left(1 + \frac{\frac{P_s}{K_s}}{\frac{P_J}{K_J} + \frac{P_N}{N_c}}\right) - \frac{\frac{P_s}{K_s}}{\frac{P_s}{K_s} + \frac{P_J}{K_J} + \frac{P_N}{N_c}} \right] \\ & + \left(1 - \frac{K_J}{N_c}\right) \frac{B}{N_c} \frac{1}{\ln 2} \left[\ln \left(1 + \frac{\frac{P_s}{K_s}}{\frac{P_N}{N_c}}\right) - \frac{\frac{P_s}{K_s}}{\frac{P_s}{K_s} + \frac{P_N}{N_c}} \right]. \end{aligned} \tag{47}$$

Let $v_1 = \frac{\frac{P_s}{K_s}}{\frac{P_J}{K_J}+\frac{P_N}{N_c}}$, then $\frac{v_1}{1+v_1} = \frac{\frac{P_s}{K_s}}{\frac{P_s}{K_s}+\frac{P_J}{K_J}+\frac{P_N}{N_c}}$. Similarly, let $v_2 = \frac{\frac{P_s}{K_s}}{\frac{P_N}{N_c}}$, then $\frac{v_2}{1+v_2} = \frac{\frac{P_s}{K_s}}{\frac{P_s}{K_s}+\frac{P_N}{N_c}}$. Applying Lemma 4 to (47), we have

$$\frac{\partial \widetilde{C}}{\partial K_s} > 0, \text{ for any } K_s = 1, 2, ..., N_c. \tag{48}$$

(2) The first-order derivative of $\widetilde{C}$ over K_J,

$$\begin{aligned}\frac{\partial \widetilde{C}}{\partial K_J} =& \frac{K_s}{N_c}\frac{B}{N_c}\frac{1}{\ln 2}\left[\ln\left(1+\frac{\frac{P_s}{K_s}}{\frac{P_J}{K_J}+\frac{P_N}{N_c}}\right) - \ln\left(1+\frac{\frac{P_s}{K_s}}{\frac{P_N}{N_c}}\right)\right. \\ &\left. + \frac{\frac{P_s}{K_s}\frac{P_J}{K_J}}{\left(\frac{P_s}{K_s}+\frac{P_J}{K_J}+\frac{P_N}{N_c}\right)\left(\frac{P_J}{K_J}+\frac{P_N}{N_c}\right)}\right] \\ <& \frac{K_s}{N_c}\frac{B}{N_c}\frac{1}{\ln 2}\left[\frac{\frac{P_s}{K_s}\frac{P_J}{K_J}}{\frac{P_N}{N_c}\left(\frac{P_s}{K_s}+\frac{P_J}{K_J}+\frac{P_N}{N_c}\right)+\frac{P_s}{K_s}\frac{P_J}{K_J}}\right. \\ &\left. - \ln\left(1+\frac{\frac{P_s}{K_s}\frac{P_J}{K_J}}{\frac{P_N}{N_c}\left(\frac{P_s}{K_s}+\frac{P_J}{K_J}+\frac{P_N}{N_c}\right)}\right)\right].\end{aligned} \tag{49}$$

Let $v_0 = \frac{\frac{P_s}{K_s}\frac{P_J}{K_J}}{\frac{P_N}{N_c}\left(\frac{P_s}{K_s}+\frac{P_J}{K_J}+\frac{P_N}{N_c}\right)}$, then $\frac{v_0}{1+v_0} = \frac{\frac{P_s}{K_s}\frac{P_J}{K_J}}{\frac{P_N}{N_c}\left(\frac{P_s}{K_s}+\frac{P_J}{K_J}+\frac{P_N}{N_c}\right)+\frac{P_s}{K_s}\frac{P_J}{K_J}}$. Applying Lemma 4 to (49), we have

$$\frac{\partial \widetilde{C}}{\partial K_J} < 0, \text{ for any } K_J = 1, 2, ..., N_c. \tag{50}$$

Appendix D
Convergence Analysis of the Iterative Water Pouring Algorithm

We prove the convergence of the iterative water pouring algorithm (please refer to Table I) by using the fact that an upper bounded and monotonically increasing sequence must converge. More specifically, we show that the constant c_2 in the algorithm is both upper bounded and monotonically increasing.

Since the total power of both the authorized user and the jammer is limited and the noise power levels are fixed, the water level, c_2, must be upper bounded. Next, we will show c_2 increases for each iteration. Recall that in the iterative water pouring algorithm, c_2 is initialized with the solution obtained using the two-step water pouring algorithm, and then we iteratively execute the following two steps:

$$\begin{cases} \frac{|H_{J,i}|^2}{|H_{s,i}|^2}P^*_{J,i} = \left[\min(\frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1, c_2) - \sigma^2_{n,i}\right]^+, \ \forall i, & \text{(51a)} \\ P^*_{s,i} = \left(c_2 - \frac{|H_{J,i}|^2}{|H_{s,i}|^2}P^*_{J,i} - \sigma^2_{n,i}\right)^+, \quad \forall i. & \text{(51b)} \end{cases}$$

In the first iteration, we resolve violations against (33) by (51a). That is, for any subchannel with unnecessarily high jamming power (i.e., $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}P^*_{J,i} + \sigma^2_{n,i} = \frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1 > c_2$), the unnecessary part $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1 - c_2$ will be moved to the less-filled subchannels (i.e., $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}P^*_{J,i} + \sigma^2_{n,i} = \frac{|H_{J,i}|^2}{|H_{s,i}|^2}c_1 < c_2$). In this case, the total jamming power below the water level c_2 goes higher, which will inevitably *raise* the water level c_2 once (51b) is executed.

Starting from the second iteration, since c_2 has been increased by the previous iteration, when reallocating the jamming power by (51a), "jamming-efficient" subchannels will be "relaxed" (due to higher c_2) in the sense of being able to use more jamming power. A subchannel is said to be more jamming-efficient, if it has a lower jamming power path loss but higher user signal path loss, i.e., the ratio $\frac{|H_{J,i}|^2}{|H_{s,i}|^2}$ is larger. For this reason, even the total original jamming power remains constant, the total "effective" jamming power below the water level c_2, which takes pass loss into account, would go higher again, due to increased contribution of highly jamming-efficient subchannels. This will again *raise* the water level c_2 once (51b) is executed. Now we are safe to say that the constant c_2 increases iteration by iteration.

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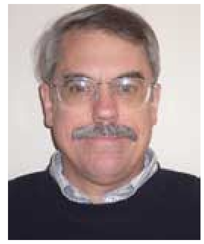

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