ABSTRACT

Computer modeling of engineering systems with a large number of interconnected multi degree of freedom (DOF) subsystems requires flexible modeling tools. Flexible modeling tools with arbitrary input-output structure formulate equations to fit the input-output structure of specific engineering systems at the cost of globally reformulating equations with every model change. Each equation formulation requires experimental performance verification. This can be particularly cumbersome in the design, development, and refinement of large models. In previous work Byam and Radcliffe designed modular modeling, a flexible modeling method that eliminates equation reformulation and enhances model experimental performance verification in large models. A fixed input-output structure for all multi DOF modular modeling elements eliminates equation reformulation. The cost is a connector constraint to assemble elements, thereby adding complexity to the global model. Solving linear modular models is a systematic realization of compatible standardized modular elements and connectors. In this work, a modular solution to linear models of engineering systems is defined. Structural and automotive examples are given.

INTRODUCTION

Design and development of today’s engineering systems rely on verified computer modeling (Jost, 1998). Computer models are becoming larger and more complex to facilitate all possible subsystem synergy within the system (“Computers In Engineering: Chrysler Designs Paperless Cars, 1998”). As the subsystems grow in number and complexity there is a greater need for systematic modeling and assembly methods to model the system while not dismissing the importance of model experimental performance verification.

There are existing modeling methods limited to one energy domain that use a systematic approach to design and development of engineering models. A systematic method of generalized Cartesian coordinates is used to design and develop kinematic and dynamic mechanical system models (Haug, 1989 and Nikravesh, 1988). A systematic method of applying Kirchhoff’s laws from a network topology is used to design and develop electrical system models (Calahan, 1972, Chua and Lin, 1975, and Vlach and Singhal, 1983). Complex engineering systems are typically hybrid with more than one-energy domains.

Finite element analysis (FEA) is a systematic modeling method that generates models in the mechanical and thermal energy domains (Zienkiewicz, 1977). Systematic grids of model elements make FEA a systematic equation generation scheme that builds up complex models. FEA model equations are generated such that independent model assembly requires reformulating an entirely new model or using special purpose software to “connect” them (PDESolve, BEAM Technologies Needham, Massachusetts), which is both cumbersome and expensive.

The bond graph approach is a power-based modeling method that generates models across many energy domains with a graphical input-output structure for easy independent model assembly (Karnopp, 1990). Graphical multi port elements and junctions make bond graphs a systematic equation generation scheme that builds up complex models. Bond graphs have arbitrary input-output definitions forcing global model reformulation per every model change. Recent bond graph research enhanced the method’s hierarchy avoiding some reformulation (Hales, 1995). Equation reformulation is the practice that prevents efficient development of large complex computer models.

Modular modeling with fixed input-output structure is a power-based systematic modeling method that eliminates equation reformulation from large model design, development, and refinement across multiple energy domains (Byam and Radcliffe, 1999). Modular modeling is a top-down systematic equation assembly scheme well suited to multi DOF components. Modeling efficiency is diminished for development and assembly of idealized single DOF model components. This method has a standardized input-output definition for all multi port multi degree of freedom modular modeling elements resulting in a single standardized element formulation. A single standardized formulation allows modelers
to gain performance verification experience enhancing the verification process. The single standardized modular modeling element combined with the compatible connector constraint makes modular model realization a simple systematic process.

**MODULAR MODELING**

Modular modeling is a modeling method designed to eliminate model equation reformulation and enhance model experimental performance verification. Modular models are constrained assemblies of power-based multi degree of freedom modular modeling elements of physical systems with a single standardized equation formulation. The key concept of modular modeling is the fixed measurement perspective input-output causality at every port of every multi degree of freedom modular modeling element per energy domain (Table 1). This causality is based on physical measurements. The port output variables are defined as the typically sensed physical system response. Development of the modular modeling method reveals that the port input variable can be assumed zero for zero power transfer and zero effect on system performance. This choice standardizes formulations and results in physical system modularity.

<table>
<thead>
<tr>
<th>Energy Domain</th>
<th>Measurement Perspective Causality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>Current Input – Potential Output</td>
</tr>
<tr>
<td>Mech. Trans</td>
<td>Force Input – Velocity Output</td>
</tr>
<tr>
<td>Mech. Rotation</td>
<td>Torque Input – Angular Velocity Output</td>
</tr>
<tr>
<td>Hydraulic</td>
<td>Volume Flow Rate Input – Pressure Output</td>
</tr>
<tr>
<td>Acoustic</td>
<td>Volume Velocity Input – Pressure Output</td>
</tr>
<tr>
<td>Heat Transfer</td>
<td>Heat Flux Input – Temperature Output</td>
</tr>
</tbody>
</table>

![Figure 1: Modular Modeling Element Graphical Notation](image)

**Table 1: Measurement Perspective Causality Across Multiple Energy Domains**

Table 1: Measurement Perspective Causality Across Multiple Energy Domains

<table>
<thead>
<tr>
<th>Energy Domain</th>
<th>Measurement Perspective Causality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>Current Input – Potential Output</td>
</tr>
<tr>
<td>Mech. Trans</td>
<td>Force Input – Velocity Output</td>
</tr>
<tr>
<td>Mech. Rotation</td>
<td>Torque Input – Angular Velocity Output</td>
</tr>
<tr>
<td>Hydraulic</td>
<td>Volume Flow Rate Input – Pressure Output</td>
</tr>
<tr>
<td>Acoustic</td>
<td>Volume Velocity Input – Pressure Output</td>
</tr>
<tr>
<td>Heat Transfer</td>
<td>Heat Flux Input – Temperature Output</td>
</tr>
</tbody>
</table>

![Figure 2: Connector Constraint Graphical Notation](image)

**Figure 1: Modular Modeling Element Graphical Notation.** Power flows into port 1 if \( u_1 \) and \( y_1 \) are both positive.

Modular modeling connector constraints implement standard output and power constraints. The connector constraint forces two connected modular element ports, \( i \) and \( j \), outputs to be equal and their power flow to sum to zero to conserve power. The power flow constraint is translated to equal and opposite inputs at connected modular element ports since the product of power port variables is power.

\[
\begin{align*}
    y_i &= y_j = y^c \\
    u_i &= u^c \\
    u_j &= -u^c 
\end{align*}
\]

(2a) The defining relationship for all connector constraints (2) does not change.

The modular modeling connector constraint graphical notation represents connectors with a bold port line between modular elements (Fig 2). By definition, connectors have the compatible input-output structure to modular elements. The bold lines have implicit standardized direction of positive power into the connected modular element ports, \( i \) and \( j \) and modular connector constraints (2). The modular connector has the flexibility to assemble by pairs any number of modular element power ports because modular modeling elements have an internal junction structure at each port (Byam and Radcliffe, 1999). The only function of modular modeling connectors is to constrain connected modular element ports.

![Figure 2: Connector Constraint Graphical Notation.](image)

**Figure 2: Connector Constraint Graphical Notation.**

The objective of modular modeling is single standardized modular formulations for all user-defined multi-port multi-degree of freedom modeling elements. This requires separate connector constraints, which adds complexity to models. However, each standardized modular modeling element has a single formulation. A single fixed formulation allows modelers to gain experience experimentally verifying their formulation’s performance. Modular models, which are constrained assemblies of \( n \) modular modeling elements, are also used with no reformulation and no reverification. Modular models are realized from a simple concatenation of \( n \) modular modeling elements.
The standardized input-output structure enables a systematic direct-insertion realization of modular models from a concatenation of modular elements and connectors.

**DIRECT-INSERTION REALIZATION OF LINEAR MODULAR MODELS**

Direct-insertion realization of linear modular models uses a concatenation of \( n \) unconstrained modular elements, a known input-output topology, and the modular modeling connector constraints (2). The user-defined linear modular element equations are linear algebraic differential equations or algebraic equations. The linear algebraic differential equations are represented in a state-space form convenient for the application of the modular modeling fixed input-output functional form. Any algebraic differential equation model can be written in form. Algebraic equation models are typically expressed in an input-output functional form.

**Linear Algebraic Differential Equations**

Modular modeling elements with user-defined linear algebraic differential equations have a traditional state-space form where the inputs and outputs are explicit.

\[
\dot{x} = Ax + Bu
\]

\[
y = Cx + Du
\]

(4)

The fixed input-output functional form (1) of modular elements is seen in (4), where \( u \) is a vector of port inputs and \( y \) is a vector of port outputs ordered in port pairs. \( x \) is a vector of states. \( A, B, C, \) and \( D \) are matrices with time invariant coefficients independent of the \( x \)-variables and \( u \)-variables.

Consider a concatenation of \( n \) independently formulated user-defined linear modular elements of the form (4). Let this concatenation have \( s \) total states and \( p \) total input-output power ports.

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\vdots \\
\dot{y}_s
\end{bmatrix} =
\begin{bmatrix}
\text{element}_1(\dot{u}_1) \\
\text{element}_2(\dot{u}_2) \\
\cdots \\
\text{element}_s(\dot{u}_s)
\end{bmatrix}
\]

The vector \( X \) is the concatenation of the \( n \) modular element state vectors with a total size \( s \times 1 \) and \( U \) is the concatenation of the \( n \) modular element input vectors with a total size \( p \times 1 \).

The fixed input-output functional form (3) of a modular model is seen in (5). Each modular element of (5) is uncoupled from the other modular elements. Given an element input-output topology the modular connector constraints (2a-2b) provide the coupling between the modular element equations.

The key concept of modular model analysis is isolating the internal element input-output power ports from the external element input-output power ports in a known input-output topology. External input-output power ports have known inputs. Internal input-output power ports are ports joined to other element ports through connectors (Fig. 2). The concatenation (5) has a total of \( p \) input-output power ports from the \( n \) modular elements. Let \( m \) be the number of connectors, hence there are \( 2m \) internal element input-output power ports, which leaves \( q = p - 2m \) external element input-output power ports. The standardized form of modular elements (1) and modular connectors (2) makes isolating external and internal element ports in (5) a simple reordering of the systems’ concatenated input and output vectors \( U \) and \( Y \).

A transformation matrix reorders the concatenated input and output vectors of (5). The vectors are reordered so all external ports’ variables appear first in the vectors followed by all the internal ports’ variables. The internal ports’ variables are ordered such that connected port pairs’ appear together. For example, if port \( i \) and port \( j \) are connected \( u_i \) should be followed immediately by \( u_j \), similarly for the outputs \( y_i \) and \( y_j \). The reordered input and output vectors are the input and output vectors of (5) pre-multiplied by the transformation matrix \( T \).

\[
TU = \begin{bmatrix}
U_{ext} \\
U_{int}
\end{bmatrix}
\]

(6a)

\[
TY = \begin{bmatrix}
Y_{ext} \\
Y_{int}
\end{bmatrix}
\]

(6b)

The transformation matrix \( T \) does not add, remove, or combine variables of the original vectors; it only changes the order in which variables appear. This makes \( T \) a linear and nonsingular reordered \( p \times p \) identity matrix \( I \).

\[
TT^T = I
\]

(7)

The external input \( U_{ext} \) is the \( q \times 1 \) vector of external port inputs. The internal input \( U_{int} \) is the \( 2m \times 1 \) vector of internal
port inputs. The external output $Y_{ext}$ is the $q \times 1$ vector of external port outputs. The internal output $Y_{int}$ is the $2m \times 1$ vector of internal port outputs. The mechanism for reordering the input and output vectors of (5) to isolate the external and internal ports is the matrix $(\mathcal{T})_{p \times q}$.

The transformation matrix $\mathcal{T}$ partitions the matrices in (5) to further isolate internal and external equations. The reordered unconnected equations are rewritten in terms of the external and internal inputs, outputs, and partitioned matrices.

$$\hat{X} = A \hat{X} + B_{ext} \hat{Y}_{ext} + B_{int} \hat{Y}_{int}$$  
$$Y_{ext} = C_{ext} \hat{X} + D_{ext} \hat{Y}_{ext} + D_{ext} \hat{U}_{int}$$  
$$Y_{int} = C_{int} \hat{X} + D_{int} \hat{Y}_{int} + D_{int} \hat{U}_{int}$$

(8a)  
(8b)  
(8c)

The internal outputs $Y_{int}$ of (8c) are ordered in connected internal port pairs. At each connected pair, the two output values are constrained by (2a) to have equal values. In order to apply the outputs of (8c) to the constraint (2a) the connected output pairs need to be selected from the matrix equation (8c). Define two $m \times 2m$ equation selector matrices $\Sigma_o$ and $\Sigma_e$ to select the odd and even equations of (8c) respectively.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}_{\text{odd}} \quad \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}_{\text{even}}$$

(9)

Pre-multiplying (8c) by $\Sigma_o$ selects the odd ($1^{st}, 3^{rd}, 5^{th}, \ldots$, $m-1^{st}$) internal port output equations. Pre-multiplying (8c) by $\Sigma_e$ selects the even ($2^{nd}, 4^{th}, 6^{th}, \ldots$, $m^{th}$) internal port output equations. Substitute selected equations in the output constraint (2a).

$$\Sigma_o Y_{int} = Y^c$$  
$$\Sigma_e Y_{int} = Y^c$$

(10a)  
(10b)

The constrained internal ports’ output $Y^c$ is a $m \times 1$ vector of the $m$ constrained inputs at the $2m$ connected internal ports (Fig. 2) of the system (5a-5b). Rewrite the output constraint (10a-10b) in terms of a difference to eliminate the constrained internal port’s output.

$$(\Sigma_o - \Sigma_e) Y_{int} = 0$$

(11)

Substituting the internal output equation (8c) into the output constraint (11) results in the output constraints of the modular model. The output constrained modular system equations are written in terms of the states, $X$, the external inputs, $U_{ext}$, the external outputs, $Y_{ext}$, and the internal inputs, $U_{int}$.

$$\begin{pmatrix}
\hat{X} \\
\hat{Y}_{ext} \\
\hat{Y}_{int} \\
\end{pmatrix} = \begin{pmatrix}
A & B_{ext} & B_{int} \\
C_{ext} & D_{ext} & D_{ext} \\
C_{int} & D_{int} & D_{int} \\
\end{pmatrix} \begin{pmatrix}
\hat{X} \\
\hat{Y}_{ext} \\
\hat{Y}_{int} \\
\end{pmatrix} + \begin{pmatrix}
0 \\
D_{ext} \hat{U}_{int} \\
D_{int} \hat{U}_{int} \\
\end{pmatrix}$$

(12a)  
(12b)  
(12c)

The three output constrained modular model equations (12a-12c) have three unknowns, the states, $X$, the external outputs, $Y_{ext}$, and the internal inputs, $U_{int}$. The objective is to find the internal input, $U_{int}$, in terms of the states, $X$, and external inputs, $Y_{ext}$, that satisfies (12c). The internal input, $U_{int}$, that satisfies the constrained output equation (12c) is found from the definition of output controllability.

$$U_{int}(\sigma) = -C^{\dagger}(\sigma \Sigma_0^{-1}(t) \Sigma_o - \Sigma_e)$$

(13)

The internal input, $U_{int}$, exists if the matrix $Y(t)$ is nonsingular. The matrix $Y(t)$ is nonsingular if it is positive definite.

$$Y(t) = \left[ \begin{array}{c}
\Sigma_o - \Sigma_e \\
\Sigma_o \delta Y(t) \end{array} \right] + \left[ \begin{array}{c}
\Sigma_o \delta Y(t) \\
\Sigma_o \delta Y(t) \end{array} \right] = \left[ \begin{array}{c}
\Sigma_o - \Sigma_e \\
\Sigma_o \delta Y(t) \end{array} \right]$$

(14a)

$$X_{int}(t_0, t) = \frac{\delta}{\delta \sigma} \left[ \begin{array}{c}
\Phi(t, \sigma) B_{int} \int_{t_0}^t C_{int} \Phi(t, \sigma) B_{ext} + D_{int} \delta(t - \sigma) \end{array} \right]$$

(14b)

The * represents the complex conjugate transpose of a matrix, $\mathcal{E}$ is arbitrarily small positive constant, $X_{int}(t_0, t)$ is the system’s “internal controllability grammian”. This derivation is similar to derivation of output controllability in Skelton, 1988. The derivation here is concerned with the controllability the internal inputs have over the external outputs or its internal output controllability. If (12a-12b) has a positive definite matrix $Y(t)$, it is internally output controllable and the internal input $U_{int}$ exists.

The condition of existence of $U_{int}$ is less restrictive using the output controllability approach of modular modeling then previous methods. Hogan found that assembling linear modular component models with simple “nonenergetic” connections required either the invertability of a matrix involving only the D matrix or that the D matrix be zero. Modular modeling clearly is less restrictive because the invertability of $Y(t)$ is dependent on the $A$, $B$, $C$, and $D$ matrices where all can be nonzero.

The internal port input pairs, $U_{int}$, are ordered the same way as the internal port output pairs, so the $m \times 2m$ equations selector matrices $\Sigma_o$ and $\Sigma_e$ can be used to define the $m \times 1$ constrained input $U^c$. The constrained input $U^c$ is the result of the modular connector power constraint (2b) constraining internal port input pairs to be equal and opposite.

$$U^c(\sigma) = \frac{1}{2} (\Sigma_o - \Sigma_e) \mathcal{E}^{\dagger}(\sigma \Sigma_0^{-1}(t) \Sigma_o - \Sigma_e)$$

(15)

The fully connected modular model equations are realized.
\[ X = A \phi + B \phi U_{ex} + B \phi \text{(16a)} \]

\[ Y_{ex} = C \phi X + D \phi U_{ex} + D \phi \text{(16b)} \]

where \( U_{ex} \) is the external input-output matrix, \( X \) is the internal input-output matrix, and \( Y_{ex} \) is the external output. The connected modular model (16a-16c) has maintained its modularity using “nonenergetic” connectors with less restrictive conditions for solution than previous methods designed to maintain modularity (Hogan, 1987).

**Linear Algebraic Equations**

Modular modeling elements with user-defined linear algebraic equations have the same form as (4) but the matrices \( A \), \( B \), and \( C \) are zero.

\[ y = Du \]  

(17)

Following the same procedure as in (5a-5b), consider an unconnected concatenation of \( n \) independently formulated user-defined modular modeling elements in the algebraic form (17) with \( p \) total input-output power ports.

\[
\begin{align*}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_n
\end{bmatrix} &=
\begin{bmatrix}
  D_{11} & 0 & \cdots & 0 \\
  0 & D_{22} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & D_{np}
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_n
\end{bmatrix}
\end{align*}
\]  

(18)

\[ Y = DU \]

The same input-output topology, reordering, and output constraint analysis applies to the algebraic output constrained modular equations. These equations are written in terms of the external inputs, \( U_{ex} \), the external outputs, \( Y_{ex} \), and the internal inputs, \( U_{int} \).

\[ (S_o - S_e)D_{int}U_{ex} + (S_o - S_e)D_{int}U_{int} = 0 \]  

(19a)

\[ Y_{ex} = DU_{ex} + D_{ext}U_{int} \]  

(19b)

Solving (19a) for internal inputs, \( U_{int} \), in terms of the \( U_{ex} \), the external outputs using the same techniques as the linear algebraic differential case (13),

\[ U_{int}(\sigma) = -D_{w1}(S_o - S_e)D_{w1}(S_o - S_e)D_{w1}(S_o - S_e)D_{w1}(U_{ex}(\sigma)) \]  

(20)

The internal input, \( U_{int} \), exists if the matrix \((S_o - S_e)D_{int}D_{int}^*(S_o - S_e)^T\) is nonsingular. This matrix is nonsingular if it is positive definite or (19a-19b) is internally output controllable. The constrained internal input, \( U^c \), for the algebraic modular system is found similar to (15),

\[ U^c(\sigma) = -\frac{1}{2}(S_o - S_e)D_{w1}(S_o - S_e)^T(S_o - S_e)D_{w1}(S_o - S_e)^T(S_o - S_e) \]  

(21)

\[ D_{w1}U_{ex}(\sigma) \]

The fully input and output constrained algebraic modular model equations are written in terms of the two unknowns constrained internal input, \( U^c \), and the external outputs, \( Y_{ex} \), where \( U^c \) is given by (21).

\[ (S_o - S_e)(S_o - S_e)^T U_{int} + (S_o - S_e)(S_o - S_e)^T U_{int} = 0 \]  

(22a)

\[ Y_{ex} = DU_{ex} + D_{ext}U_{int} \]  

(22b)

**Linear Algebraic Boundary Value Problem Equations**

The linear algebraic equations that result from a boundary value problem (BVP) have the form of the generalized responses, \( x \), pre-multiplied by a stiffness matrix, \( K \), equal to the generalized excitations, \( \omega \) (Segerlind, 1984).

\[ Kx = \omega \]  

(23)

Without boundary conditions the stiffness matrix, \( K \), is singular. These equations occur in structural, solid mechanics, heat transfer, and irrotational flow models typically developed by Finite Element Analysis (FEA). Consider a concatenation of \( n \) of these modeling elements without boundary conditions applied.

\[
\begin{bmatrix}
  D_1 & 0 & \cdots & 0 \\
  0 & D_{22} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & D_{np}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix} =
\begin{bmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_n
\end{bmatrix}
\]  

(24)

\[ KX = W \]

The generalized responses, \( X \), and the generalized excitations, \( W \), are defined by element geometry which differ for each of the \( n \) modeling elements. In order to maintain modularity, the \( n \) modeling elements in (24) will be connected through inputs, \( U \), and outputs, \( Y \), that are linear interpolations of the generalized responses, \( X \), and the generalized excitations, \( W \).

\[
\begin{bmatrix}
  D_1 & 0 & \cdots & 0 \\
  0 & D_{22} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & D_{np}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix} =
\begin{bmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_n
\end{bmatrix}
\]  

(25a)

\[ KX = WC^T U \]  

(25b)

Notice the input-output matrix \( C \) is the same in both (25a) and (25b). This is because the co-located inputs and outputs are defined as ports. BVP modeling elements in this form can be connected together maintaining modularity.

The input-output topology, reordering scheme, and input-output constraint analysis applied to the modular BVP system equations (25a-25b). These fully connected equations are
written in terms of the external inputs, \( U_{ext} \), the external outputs, \( Y_{ext} \), and the constrained internal inputs, \( U^c \), using the equation selector matrices, \( \mathbf{S}_o \) and \( \mathbf{S}_e \).

\[
\mathbf{K} \mathbf{X} = \mathbf{C}_{\text{ext}}^T U_{\text{ext}} + \mathbf{C}_{\text{int}}^T (\mathbf{S}_o - \mathbf{S}_e)^T U^c
\]  
\[26a\]
\[
Y_{\text{ext}} = \mathbf{C}_{\text{ext}} \mathbf{X}
\]  
\[26b\]
\[
(\mathbf{S}_o - \mathbf{S}_e) \mathbf{C}_{\text{int}} Y = \mathbf{U}_{\text{ext}}
\]  
\[26c\]

This realization is different then the state equations (16a-16c) and (22a-22b) because the stiffness matrix \( \mathbf{K} \) is singular prior to application of boundary conditions.

Solving (26a-26c) for the generalized responses, \( \mathbf{X} \), the constrained inputs, \( \mathbf{U}^c \), and the external outputs, \( \mathbf{Y}_{\text{ext}} \), in terms of the external inputs, \( \mathbf{U}_{\text{ext}} \) requires finding a set of generalized responses that satisfy the output constraints (26c). Any basis of the nullspace of \( (\mathbf{S}_o - \mathbf{S}_e) \mathbf{C}_{\text{int}} \) will define a set of output constrained generalized responses that satisfy (26c). The nullspace of \( (\mathbf{S}_o - \mathbf{S}_e) \mathbf{C}_{\text{int}} \) has dimension \( s - m \) because it has rank of \( m \) (Leon, 1986). It has rank of \( m \) because it represents \( m \) connectors at \( m \) different physical locations resulting in \( m \) independent rows seen in the structure of \( (\mathbf{S}_o - \mathbf{S}_e) \). Define a \( s \times s - m \) matrix, \( \mathbf{V} \), that is a basis of the nullspace of \( (\mathbf{S}_o - \mathbf{S}_e) \mathbf{C}_{\text{int}} \) to transform the output constrained generalized responses, \( \mathbf{X}^c \), to the generalized responses, \( \mathbf{X} \).

\[
\mathbf{X} = \mathbf{V} \mathbf{X}^c
\]  
\[27\]

This procedure is similar to the procedure Meirovitch uses to eliminate rigid body modes (Meirovitch, 1967). In Meirovitch a basis is the rigid body mode eigenvectors. Substitute (27) into (26a) and pre-multiply by \( \mathbf{V}^T \) with the knowledge that \( \mathbf{V}^T \mathbf{C}_{\text{int}}^T (\mathbf{S}_o - \mathbf{S}_e)^T = 0 \) because of the orthogonality of nullspaces (Leon, 1986).

\[
\mathbf{V}^T \mathbf{K} \mathbf{V} \mathbf{X}^c = \mathbf{V}^T \mathbf{C}_{\text{ext}}^T \mathbf{U}_{\text{ext}}
\]  
\[28\]

The matrix \( \mathbf{V}^T \mathbf{K} \mathbf{V} \) has dimension \( s - m \times s - m \). The matrix \( \mathbf{V} \) combines rows and columns of the system stiffness matrix \( \mathbf{K} \) connecting the otherwise independent element stiffness matrices \( \mathbf{K}_i \). This pre and post multiplication has the same effect as applying the direct stiffness method (Segerlind, 1984). Solving for \( \mathbf{X}^c \) requires the matrix \( \mathbf{V}^T \mathbf{K} \mathbf{V} \) to be nonsingular. This requires the elimination of all rigid body modes. The solution of (28) \( \mathbf{X}^c \) can be used to find the external outputs \( \mathbf{Y}_{\text{ext}} \) and the constrained internal inputs \( \mathbf{U}^c \).

The linear algebraic differential, linear algebraic, and linear BVP constrained modular model equations (16a-16c), (22a-22b), and (26a-26c) respectively are written in terms of independently formulated modular modeling elements. The reordering and selector matrices, \( \mathbf{T} \) and \( (\mathbf{S}_o - \mathbf{S}_e) \), are comprised of ones and zeros and defined by the size and input-output topology of the modular model. The modular modeling element matrices \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \) and \( \mathbf{K} \) can be directly inserted into the constrained modular model equations and solved like any other system of equations. The advantage of modular modeling is that each multi DOF modeling element is independent and can be connected together with simple connectors and assembled with a simple physically intuitive top-down direct insertion method.

**Examples**

An electric automotive vehicle power train example model is used to illustrate modular modeling of linear algebraic differential modular models and linear algebraic modular models. A structural three-bar and connecting shear pin example model is used to illustrate modular modeling of linear algebraic BVP modular model. The electric automotive vehicle power train example model is a combination of three simple dynamic models found in Phillips and Harbor, 1996 and Minor, 1996. The three bar and connecting shear pin example model is a combination of two simple BVP models from Segerlind, 1984.

The electric automotive vehicle power train example model contains three simple modular modeling elements, an electric motor model, a clutch model, and a transmission model. The three modular modeling elements are all defined with standardized modular modeling causality (Table 1). The electric motor modular modeling element has one electrical power port with current input \( i \) voltage output \( e \) causality and one rotational mechanical power port with torque input \( \tau \) angular velocity output \( \omega \) causality (Fig. 3a).

\[
\begin{bmatrix}
\dot{x}_m \\
\dot{\theta}_m
\end{bmatrix}
= \begin{bmatrix} K_t & Jt \\
1 & 0
\end{bmatrix} \begin{bmatrix} \tau \\
\omega
\end{bmatrix}
\]  
\[29a\]
\[
\begin{bmatrix}
e \\
\omega
\end{bmatrix}
= \begin{bmatrix} K_t & Jt \\
1 & 0
\end{bmatrix} \begin{bmatrix} d \tau \\
d\omega
\end{bmatrix}
\]  
\[29b\]

The motor model has a state, \( x_m \), a lumped motor-shaft rotational inertia, \( J_t \), a motor coil resistance, \( R \), a motor back emf constant, \( K_t \), and a motor torque constant, \( K_e \). The transmission modular modeling element has two rotational mechanical power ports with torque \( \tau_m \) and \( \tau_i \) input angular velocity \( \omega_m \) and \( \omega_t \) output causality (Fig. 3b).

\[
\begin{bmatrix}
\dot{x}_m \\
\dot{x}_t
\end{bmatrix}
= \begin{bmatrix} c_{\tau m} & GR \\
J_{\tau m} & J_{\tau t}
\end{bmatrix} \begin{bmatrix} \tau_m \\
\tau_i
\end{bmatrix}
\]  
\[30a\]
\[
\begin{bmatrix}
\omega_m \\
\omega_t
\end{bmatrix}
= \begin{bmatrix} c_{\tau m} & GR \\
J_{\tau m} & J_{\tau t}
\end{bmatrix} \begin{bmatrix} \tau_m \\
\tau_i
\end{bmatrix}
\]  
\[30b\]

The transmission model a state, \( x_m \), a lumped transmission rotational inertia, \( J_{\tau m} \), an equivalent transmission damping, \( c_{\tau m} \), and a transmission gear ratio, \( GR \). The clutch modular modeling element has two rotational mechanical power ports with torque \( \tau_m \) and \( \tau_i \) input angular velocity \( \omega_m \) and \( \omega_t \) output causality (Fig. 3c).
The clutch model has three states, $[x_{c1}, x_{c2}, x_{c3}]^T$, a lumped clutch rotational inertia, $J_{cl}$, and an equivalent torsional stiffness, $K_{cl}$.

The input-output topology of the two element power train modular model defines the size and structure of the modular modeling reordering and equation selector matrix $\mathbf{T}$ and $(\mathbf{S}_o - \mathbf{S}_e)$ respectively. This modular model has four power ports and one connector. This results in a $4 \times 4$ reordering matrix $\mathbf{T}$ and a $1 \times 2$ equation selector matrix $(\mathbf{S}_o - \mathbf{S}_e)$.  

$$
\mathbf{T} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad (33)
$$

Ports 1 and 4 are the external power ports so $\mathbf{T}$ reorders the input $U = [i, \tau, \tau_u, \tau_d]^T$ and output $Y = [e, \omega, \omega_u, \omega_d]^T$ so $i$, $\tau_d$, $e$, and $\omega_d$ appear first in the vectors. Ports 2 and 3 are connected internal power ports reordered to appear in connected pairs after the external ports. Apply the reordering matrix $\mathbf{T}$ to (32a-32b) to partition the inputs and outputs internally and externally.

Apply the internal input and output constraints (2a-2b) to the two element electric power train modular model (Fig. 4) and rewrite in the fully connected modular model (16a-16c).

The unconnected two element modular system is built up from directly inserting the motor and transmission modular element matrices into (5a-5b).
The unconnected three element modular system is built up from directly inserting the motor, clutch, and transmission modular element matrices into (5a-5b).

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -K_j & 0 & 0 & 0 \\
0 & 0 & -J_j & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -C_r & 0
\end{bmatrix}
\begin{bmatrix}
x_m \\
x_cl \\
x_m \\
x_cl \\
x_r
\end{bmatrix}
= \mathbf{1}
\begin{bmatrix}
x_m \\
x_cl \\
x_m \\
x_cl \\
x_r
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\gamma \\
\delta \\
\epsilon
\end{bmatrix}
\]

Clearly, \(C_m\) has a rank of 4, which is the number of internal outputs. The internal input \(u_{32} \in \mathbb{R}^4\) exists and the modular 3 element electric vehicle model (37a-37c) can be solved like any other state system model.

The linear algebraic modular model example equations is defined by taking the Laplace transform of the two element electric automotive vehicle power train modular model equations (32a-32b). Form of the unconnected linear algebraic modular model (18).

\[
Y = DU = \mathbf{1}
\begin{bmatrix}
\frac{s}{s+K/J} & \frac{s}{s+K/J} \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
u \\
\omega_m \\
\omega_c \\
\omega_r
\end{bmatrix}
\]

The two element linear algebraic electric vehicle power train modular model (39) has the same input-output topology (Fig. 4) as the linear algebraic differential system (32a-32b). So, (39) can be reordered and constrained with the \(\mathcal{T}\) and \((\mathcal{S}_s - \mathcal{S}_r)\) in (33) to obtain the fully connected linear algebraic modular form (22a-22b).

\[
(\mathcal{S}_s - \mathcal{D})_{\text{Initial}} U_{\text{Initial}} + (\mathcal{S}_s - \mathcal{S}_r)Y = 0
\]

The internal input \(u_{32} \in \mathbb{R}^4\) exists if (40a-40b) is internally output controllable or the matrix

\[
(\mathcal{S}_s - \mathcal{S}_r) \mathcal{D}_{\text{Initial}} (\mathcal{S}_s - \mathcal{S}_r)^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The internal input \(u_{32} \in \mathbb{R}^4\) exists and the modular linear algebraic two element electric vehicle power train model (40a-40b) can be solved like any other linear algebraic system model. A similar analysis can be done for the modular three element electric vehicle power train model.

The linear algebraic BVP modular model example is a structural BVP model with three bars and connecting shear pin fixed to a point. This structural modular BVP model example contains two simple modular modeling elements, an axial force bar model, a bending pin model, and a fixed point model. The axial force bar modular modeling element is two node Finite Element Model (FEA) from Segerlind, 1984. This model has two mechanical ports with force input \(F\) axial displacement output \(\delta\) causality (Fig. 6a), which is written in linear algebraic BVP modular modeling form (25a-25b).
The modular bar BVP (42a-42b) has two states \([x_a, x_b]^T\), a cross-sectional area \(A\), a modulus of elasticity \(E\), and a length \(L\).

The fixed point modular modeling element is unique to modular modeling because of the standardized measurement perspective causality. The one port fixed point modular modeling element outputs a zero displacement regardless of the input and applies to any energy domain (Fig. 6b). The fixed point modular modeling element is written in linear algebraic BVP modular modeling form (25a-25b).

\[
\begin{align*}
K x &= C^T u \\
AE \begin{bmatrix} 1 & -1 \end{bmatrix} x_a &= \begin{bmatrix} 1 & 0 \end{bmatrix} F_i \\
y &= C x \\
\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_a \\
\end{align*}
\]

(42a, 42b)

The first two rows of \(T\) identify the external ports and the remaining six rows order the internal ports in connected pairs.

Figure 6: Two Linear Algebraic BVP Modular Modeling Elements: a) Axial Force Bar b) Fixed Point

An example linear algebraic BVP modular models will use the two modular modeling elements (Fig. 6). Three axial force bars and a fixed point will be connected together in a bracket-pin-clevis configuration with no pin bending deflection (Fig. 7).

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{bar}_1} \\ x_{\text{bar}_2} \\ x_{\text{bar}_3} \end{bmatrix} + \begin{bmatrix} x_{\text{fix pt}} \\ x_{\text{fix pt}} \end{bmatrix}
\]

(45a)

\[
\begin{bmatrix} y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\text{bar}_1} \\ x_{\text{bar}_2} \\ x_{\text{bar}_3} \end{bmatrix}
\]

(45b)

\[
\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{bar}_1} \\ x_{\text{bar}_2} \\ x_{\text{bar}_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

(45c)

Find the basis for the nullspace of \((S_o - S_e)\).

\[
V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

(46)

At this point the linear algebraic BVP modular system (45a-45c) can be solved like any other linear algebraic BVP model.
CONCLUSIONS

Modular modeling reduces the experimental verification task of large model design, development, and refinement by standardizing the functional form of all multi degree of freedom modeling elements. Assembling incompatible modular modeling elements requires a 2-port connector constraint which adds to model complexity. The separate modular modeling elements and connectors enable subsystem level modeling with no reformulation and no experimental reverification.

The realization of linear modular models is a systematic direct insertion procedure. Modular modeling elements with nonlinear equations are the subject of a future paper. Given n modular modeling elements and an input-output topology of their interconnections a systematic reordering and constraint analysis realizes a built up modular model that can be solved like any other linear system of equations. Modular modeling elements maintain their modularity in the assembled modular model. This method’s maintenance of element modularity while assembling with simple connectors with a less restrictive condition on the modular elements is previously unavailable in other methods.

REFERENCES


Haug, E. J., Computer-Aided Kinematics And Dynamics of Mechanical Systems: Volume I: Basic Methods, Allyn and Bacon, Massachusetts, 1989


PDEsolve, BEAM Technologies, 687 Highland Avenue, Needham, Massachusetts


