Active Parametric Damping of Distributed Parameter Beam Transverse Vibration

An active vibration control for a modified, nonlinear, dynamic, simply supported, Bernoulli-Euler beam is introduced using one of the distributed, time-dependent parameters of the system. The control is carried out by observing the axial velocity of the end point of the beam and applying a modified bang-bang variation of beam tensile stress to control beam transverse stiffness. Numerical simulation of the closed-loop system of partial differential equations demonstrates the effectiveness of the control. Two cases representing initial value problems are given as examples. This active control applied to first mode vibration of an undamped system model yields an asymptotically stable system which loses less than 0.26 percent of its initial energy in five and one half cycles.

Introduction

The vibration of a distributed parameter system (DPS) is governed by one or more coupled partial differential equations (PDE's) (Metivovich, L., 1967) whose coefficients or parameters are, in general, functions of spatial variables and time. Three current approaches to the active control of vibration in distributed structures include: Modal Control (Balas, 1978, 1982; Metivovich and Baruh, 1981), Spectral Control (Radcliffe and Mote, 1983), and Distributed Parameter Feedback (Burkovski, 1969, and Kohnle, 1978). Modal Control uses a finite number of modes to approximate the dynamics of a vibrating distributed parameter structure. A spacial description of each mode, e.g., its eigenfunction, in this finite set is used to separate the total motion of the system into the components of each of these modes. Because motion in modes not included in the modeled modes always occurs, this motion results in truncation errors in the observation algorithm referred to as "observation spillover" (Balas, 1982). Spectral Control separates the motions of the modes using their different natural frequencies, e.g., eigenvalues, instead of the eigenfunctions used in Modal Control. This method also suffers from spillover problems associated with modes with repeated eigenvalues or at higher eigenvalues than those modeled. Spacial filtering has been employed with this method to reduce frequency domain spillover and improve controller performance. In Distributed Parameter Feedback Control, the system model retains an infinite number of modes. To date there is theoretical work with few applications in the literature (Bailly and Hubbard, 1983).

Active Parametric Control discussed here uses controlled parameter fluctuation to reduce structural vibration. It is well known that transverse and longitudinal vibrations of a beam are coupled (Lubkin and Stoker, 1943).

If a transversely vibrating beam is subjected to time dependent force, $P(t)$ (Fig. 1), at its moving boundary, a parametric, time varying force, $p(x,t)$ will be produced in the beam. If the force, $P(t)$, is applied with an appropriate control algorithm, the induced parametric force, $p(x,t)$, can be used to reduce transverse vibration. Bailly et al. (1979, 1982) studied the abstract problem of controlling a semi-linear evolution equation and applied the formalism to the case of a Bernoulli-Euler beam with parametric force, $p(x,t) = P(t)$. They proved the control-ability of a finite number of modes for finite observations, $y_i(t)$, and $\partial y_i / \partial t$, where $i=1,N$ and $N$ is the number of sensor locations. They required that the initial conditions be active in all modeled modes.

In this paper, the vibration control of a modified, nonlinear, dynamic, Bernoulli-Euler beam is studied. The direct method of Liapunov is applied to develop a control algorithm yielding asymptotic stability. The control enables one to map from observing and controlling a theoretically infinite number of points in the domain into observing and controlling a single point to stabilize the system. To demonstrate the effectiveness of the derived control algorithm, a numerical finite-difference simulation of the closed-loop control is used for two examples.

![Schematic diagram of a beam under parametric control](image-url)
Theoretical Analysis

Mathematical Model. The governing equations of motion for transverse displacement, y = y(x,t), axial displacement, u = u(x,t), and parametric force, p = p(x,t), of a modified, nonlinear, dynamic, simply-supported, constant cross-section Bernoulli-Euler beam in a plane are given by

\[ EJ_{xx}'' - p_{yy} = \frac{\rho A}{2} \left( \frac{d^2y}{dx^2} + \frac{du}{dt} \right)^2 \quad (1) \]

where \( f(x,t) \) is the external transverse force distribution on the beam, and beam parameters are mass density, \( \rho \), moment of inertia, \( I \), modulus of elasticity, \( E \), cross-sectional area, \( A \), and length, \( L \). The subscript notation \( y = \frac{dy}{dx}, u = \frac{du}{dt}, \) etc. is used here. The axial strain-displacement relationship is (Sevin, 1960 and Habib, 1988).

\[ \rho /EA = u_t + (1/2)\ddot{u}_t \quad (3) \]

with the boundary conditions

\[ y(0,t) = y(L,t) = 0 \quad (4a) \]

\[ y_{xx}(0,t) = y_{xx}(L,t) = 0 \quad (4b) \]

\[ u(0,t) = 0 \quad (5) \]

\[ EA \left[ u_t(L,t) + (1/2)\ddot{u}_t(L,t) \right] = P(t) \quad (6) \]

and the initial conditions

\[ y(x,0) = f_1(x), \quad 0 < x < L \quad (7a) \]

\[ y(x,0) = f_2(x), \quad 0 < x < L \quad (7b) \]

\[ u(x,0) = 0 \quad (8a) \]

\[ u(x,0) = g(x) \quad (8b) \]

where \( f_1(x) \) and \( f_2(x) \) are initial transverse displacement and velocities respectively, and \( g(x) \) is the initial longitudinal displacement of the beam.

The above mathematical model assumes uniform, linearly elastic material properties and uniform beam cross section. The effects of shear deformation, rotary inertia and passive damping are not considered for simplicity and to emphasize the control concept.

Stability Analysis. Active Parametric Control theory is based on using one of the time dependent distributed parameters to control transverse displacement. Equation (1) contains the parametric axial tension, \( p(x,t) \), as a coefficient. The control objective is to use \( p(x,t) \) to form an asymptotically stable feedback-loop system. We will define a Liapunov functional from the total energy of the system and derive a control law for \( p(x,t) \) which guarantees asymptotic stability. Zubov (1964) and Wang (1966, 1967) extended the Liapunov stability theory from finite dimensional systems to infinite dimensional systems and the realm of distributed parameter systems. As with finite dimensional systems, the main difficulty (Lapidus and Berger, 1968) is identification of an appropriate Liapunov functional, \( V \).

Leibholz (1969) showed the close connection between Liapunov's stability criterion and the classical energy criterion expressed via the Hamiltonian, \( H \), for autonomous, dynamic, continuous systems. He proved that for a conservative system, if \( V \) is chosen as the Hamiltonian, \( H \), then

\[ \frac{dV}{dt} = 0 \quad (9) \]

and for a nonconservative system

\[ \frac{dV}{dt} = \int_0^t Q \, dq \, dV_0 \quad (10) \]

where \( Q \) is the vector of generalized forces, \( q \) is the generalized coordinate vector, and \( V_0 \) is the volume of the system. Here the control \( V \) as the total energy of the beam system for transverse force distribution, \( f(x,t) = 0 \).

\[ V(q) = T - W = \int_0^t \left[ \frac{\rho A}{2} \left( \frac{d^2y}{dx^2} + \frac{du}{dt} \right)^2 + E \int_0^L \left( \frac{d^2y}{dx^2} \right)^2 + p^2/EA \right] dx \quad (11) \]

here generalized coordinate vector, \( q^T = [y_t, \gamma_t, p] \). We state that \( q = 0 \) corresponds to the equilibrium state of the beam. At \( q = 0 \), beam transverse displacement, \( y = y(x,t), 0 \), because \( y(0,0) = y(L,0) = 0 \). Using (3), \( y(x,t) = p(x,t) = 0 \) yields \( u_t(x,t) = 0 \). This condition combined with boundary conditions (5) and (6) yield the equilibrium \( y(x,t) = u(x,t) = 0 \) at \( P(t) = 0 \) with displacement time derivatives, \( y_t = u_t = 0 \). The sign of \( V(q) \), and its time derivative will be investigated (Dym, 1974).

Define the auxiliary vectors, \( \xi = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5] \) and \( \delta = [\xi_6, \xi_7, \xi_8, \xi_9] \) so that we can introduce the metric, \( \rho_1(\xi, \delta) \) as

\[ \rho_1(\xi, \delta) = \left( \frac{1}{2} \right) \int_0^L \left[ \rho A \xi_1^2 + \rho A \xi_2^2 + \rho A \xi_3^2 + \rho A \xi_4^2 + \rho A \xi_5^2 + \rho A \xi_6^2 \right] dx \quad (12) \]

It follows that

\[ \rho_1(q,0) = \left( \frac{1}{2} \right) \int_0^L \left[ \rho A y_t^2 + \rho A u_t^2 + E \int_0^L \left( \frac{d^2y}{dx^2} \right)^2 dx \right] dx \quad (13) \]

Thus \( \rho_1(q,0) \) is a measure of the distance between the equilibrium state, \( q = 0 \), and the deformed state, \( q \neq 0 \). Further, if \( \rho_1(q,0) \) is small, then each of the terms,

\[ \int_0^L \rho A y_t^2 dx \quad \int_0^L \rho A u_t^2 dx \quad \int_0^L E \int_0^L \left( \frac{d^2y}{dx^2} \right)^2 dx \]

must be small because of the integrand of \( \rho_1(q,0) \) is the sum of these non-negative terms. It is clear that the metric \( \rho_1(\xi, \delta) \) is positive definite, symmetric and obeys the triangle inequality (Friedman, 1966).

To prove the stability of the equilibrium state \( q = 0 \), and find conditions for stable control force, \( P(t) \), we must show that:

(a) \( V(q) \) the positive definite with respect to metric \( \rho_1(q,0) \).

(b) \( V(q) \) admits an infinity small upper bound in the neighborhood of \( q = 0 \), and

(c) \( dV(q)/dt \) is negative definite.

Comparing (11) with (13) yields \( V(q) \geq \rho_1(q,0) > 0 \) for \( q \neq 0 \) therefore \( V(q) \) is positive definite. To prove that \( V(q) \) admits an infinity small upper bound in the neighborhood of \( q = 0 \) requires that \( V(q) \leq \gamma \rho_1(q,0) \) where \( \gamma \) is a positive constant and \( \rho_1 \) is the metric defined by

\[ \rho_1(q,0) \]

(14)

This condition assures continuity of the solutions to the partial differential equations, (1–6), with respect to the initial conditions, (7,8). Choosing any \( \gamma > 1 \) satisfies condition (b) since \( \rho_1(q,0) = V(q) \).

The time derivative of \( V(q) \) from (11) is

\[ \frac{dV}{dt} = \int_0^t \left[ \rho A y_t y_{tt} + \rho A u_t u_{tt} + E \int_0^L \frac{d^2y}{dx^2} + p^2/EA \right] dx \quad (15) \]

Using the constitutive relationship (3) and rearranging terms yields
\[ \frac{dV(q)}{dt} = \int_0^L \left( \rho A u_x^2 \right) dx + \int_0^L \left( \int_0^L \rho A u_x u_y dx \right) dx \
+ \int_0^L \left( E I \frac{d^2}{dx^2} + \rho \frac{d^2 u_x}{dx^2} \right) dx \]  
(16)

Integrating by parts and applying the boundary conditions (4-6) yields

\[ \frac{dV(q)}{dt} = \int_0^L \left( \frac{d^2}{dx^2} \left( u_x^2 \right) - \rho u_x^2 \right) dx + \rho A u_x u_y \]  
\[ \int_0^L \left( A u_{xx} - p_x \right) u_x dx + p u_x \]  
(17)

Using the equations of motion (1-2) and noting that \( u_x(0,t) = 0 \) yields the stability condition

\[ \frac{dV(q)}{dt} = p(L,t) u_x(L,t) = P(t) u_x(L,t) \leq 0 \]  
(18)

for asymptotic stability where \( u_x(L,t) \) is the beam end point velocity and \( P(t) \) is the axial force applied at that end. In this case, \( dV(q)/dt \) can be negative definite if the space where \( dV(q)/dt \) contains no nontrivial trajectories of the system (Henry, 1981). For the beam considered here, the only nontrivial trajectory where \( u_x(L,t) = 0 \) is the system equilibrium at \( q = 0 \).

The sign of the \( P(t) u_x(L,t) \) product is not known in general, however, this condition can be used as the foundation for an asymptotically stable control algorithm. Any active control algorithm given by

\[ P(t) - g(u_x(L,t)) \]  
(19)

where \( g \) is any strictly negative function of \( u_x(L,t) \) will yield asymptotic stability. In the \( P(t) - u_x(L,t) \) plane, any function residing strictly in the second and fourth quadrants such as the control force shown in Fig. 2 satisfies this condition.

**Simulation Results**

To test the control algorithm, a simulation was programmed on a Prime 750 minicomputer in the A.H. Case Center for Computer-Aided Engineering and Manufacturing at Michigan State University. The beam axial motion (2) was solved analytically in closed form using Finite Sine Fourier Transforms. The beam transverse motion was approximated using an explicit finite difference scheme (Wang, 1967) using a mesh of \( N = 15 \) nodes along the length of the beam and convergence parameter, \( \Delta t / \Delta x^2 = 0.219 \) s/m². The simulation was performed using beam properties and dimensions corresponding to a laboratory prototype under design (Table 1).

Two test cases will be presented to illustrate the effectiveness of the modified bang-bang active control. Other cases have been investigated (Habib, 1988). The two cases were selected to examine transient response due to non-zero initial conditions in only the lowest frequency transverse beam bending mode, and then for non-zero initial conditions in many beam transverse bending modes. A modified bang-bang control force (Fig. 2) was used with control force magnitude, \( \alpha \), varied and \( c = 3 \times 10^{-5} \) m/sec for all test cases discussed here.

**Initial Displacement of Lowest Frequency Bending Mode**

The beam displacement initial condition functions were specified with the shape of the lowest frequency beam transverse bending mode, \( f_1 = y(x,0) = 1.0 \sin \left( \pi x / L \right) \), \( f_2 = y(x,0) = 0 \), and \( g_1 = u(x,0) = 0 \) in order that only that mode was excited. Figure 4 shows the transverse displacement of the middle of the beam, \( y(L/2,t) \) versus time for the active control force magnitudes, \( \alpha = 0, 40, 80 \), and 108 N.

The free response for the uncontrolled beam, \( \alpha = 0 \), has a constant magnitude at the first transverse natural frequency of the beam, \( \omega = 9.65 \) Hz. The responses for \( \alpha = 40 \) and 80 are oscillatory and decay more rapidly with increasing control force magnitude, \( \alpha \). When the control force, \( \alpha \), is increased to 108 N, the response is no longer oscillatory and the beam transverse displacement decays monotonically.

The control forces generated in the first test case are shown in Fig. 5. The saturating nature of the bang-bang control dominates the character of the control forces while beam transverse velocity, and hence beam end velocity, \( u_x(L,t) \) is large. Once the end velocity decays to less than \( c \) (Fig. 2), linear control action results and the control force decays exponentially to zero with beam transverse vibration. The beam end velocity,
u(t), for control, α = 108 N never decays to t in the time period shown and this control remains in saturation.

The natural frequencies of the beam are changed by the control force, $P(t)$, which changes beam tensile force parameter, $p(x,t)$. This effect can be seen in the different shapes of the response wave forms for positive and negative $P(t)$. The effect is especially apparent at $\alpha = 80$ in the Fig. 4. For the first quarter of the first oscillation, $y(L/2,t) > 0$ and $y(L/2,t) < 0$, beam end velocity, $u_t(L,t)$ is greater than zero resulting in a compressive control force, $P(t)$ which reduces beam effective stiffness, lowers beam natural frequencies and increases the time for the first quarter oscillation period. For the second quarter period with $y(L/2,t) > 0$ and $y(L/2,t) < 0$, the beam end velocity, $u_t(L,t)$ is less than zero resulting in a tensile control force, $P(t)$ which increases beam natural frequencies and decreases the time for this quarter of the oscillation period. Greater control force magnitude, $\alpha$, increases the magnitudes of these changes in frequencies. For $\alpha = 40$ N, the beam lowest natural frequency fluctuates between 7.3 Hz and 11.6 Hz based on simulation results (Habib, 1988).

The rapid decay of transverse displacement magnitude for controls, $\alpha = 40$ and 80 N and the slower decay for control, $\alpha = 108$ N suggests that an optimal control force magnitude could be defined. In this example, $\alpha = 80$ N reduces vibration amplitude most rapidly.

**Initial Displacement of Many Beam Bending Modes.** A test where many beam bending modes were excited was generated by selecting an initial displacement which obeyed beam boundary conditions while not having the shape of any single transverse bending mode: $f_1 = y(x,0) = 2.15 \times (x - L)$ mm where $x$ is measured in meters $f_2 = y(x,0) = 0$, and $g_1 = u(x,0) = 0$. Figure 6 shows the transverse displacement of the middle of the beam, $y(L/2,t)$ versus time for the active control force magnitude, $\alpha = 0$, 10 and 30 N.

The second test’s initial displacement of the beam yielded a free response, $\alpha = 0$ with multiple excited modes dominated by the first bending mode. Figure 6 shows that this multimode vibration is reduced at increasing decay rates as the control, $\alpha$ is increased. The saturation characteristic of the control is apparent again for this test case (Fig. 7). In this case, however, the clear linear control region at lower displacement amplitudes is missing because of the high frequency components in the controlled response and corresponding higher beam end points velocities. These higher end point velocities are seen in the results as control force “impulses” at ever decreasing frequency as the end point velocities are reduced by the control action.

**Conclusion**

Active Parametric Control has been shown to be an efficient technique for the control of simply-supported beam transverse vibration. An energy integral was derived from the beam's coupled partial differential equations which was used to directly derive a Liapunov functional. This Liapunov functional was then used to deduce a stabilizing, closed-loop, parametric control for the beam. The closed-loop hyperbolic equation for beam axial displacement was solved using the finite sine Fourier transform and the closed-loop parabolic equation for beam transverse displacement was solved using an explicit finite difference approximation. Using beam geometry and material properties corresponding to those of anticipated laboratory tests, the parametric control was tested numerically and two numerical tests with different initial conditions presented here. In both cases, the proposed control was able to damp transverse vibration with the authority required to produce overdamped response. In these test cases, the control was able to reduce vibration amplitude to 10 percent of its original magnitude in times as short as that required for 1.5 undamped cycles.

**References**

