TOPOLOGY OPTIMIZATION OF HEAT-RESISTANT STRUCTURES

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ABSTRACT

The standard problem of finding the optimal layout of structural material associated with maximum stiffness is expanded to include consideration of thermal criteria. The problem is posed as a three-phase layout problem where the phases include an insulating or fire retardant material and an unknown distribution of heat sources, in addition to the structural material. The model used is simple, yet results suggest that the introduction of measures to control the temperature in the structure when subjected to significant heat transfer rates can result in layouts that differ substantially from solutions where thermal issues are ignored.

Keywords: multiphysics topology design optimization, topology optimization, layout optimization.

INTRODUCTION

The topic of interest in this work is the design and protection of structures exposed to a significant heat transfer rate such as during a fire. It is assumed here that the heat created by this fire is large and can damage the structure. Thus, design considerations should include measures to protect the structure from heat, in addition to the more common criteria related to structural performance, such as stiffness or strength. The problem is cast in the fashion of a standard topology optimization problem, where the shape of the structure is represented as a material property $1,2$. However, in this case, in addition to the layout of structural material, the optimization problem seeks optimal distributions of fire protective material that surrounds but does not stiffen the structure.

Fires typically involve two steps: ignition and flame. Flame propagation, which often follows ignition, occurs by heating of the adjacent materials to the point where the volatiles released by the degradation of the solid materials can be used as fuel. The mechanisms associated with the release of the volatiles are quite complex and involve two distinct processes: fuel generation in the solid phase and combustion in the gas phase. Radiation from the flame, conduction or convection through the gas from the flame and conduction through the solid are three important mechanisms -besides external sources- that control the rate of flame spread.

Modeling of flame propagation is very complex. As a first approximation, the presence of a fire and its effect on a structure can be very crudely approximated by replacing the burning medium with a medium with a large uniform heat source. This would represent, for example, a fire that occupies all the available space but does not burn the insulation. The effects of radiation and convection are neglected initially and replaced by a conduction heat transfer process with heat source. The effect of the heat source is to create temperature profiles in the medium that approximate the temperature profiles of a heated medium with convection. This is the extent of the thermal analysis performed here. In addition, the effect of rising temperatures on the structure is dealt with only -implicitly-, by constraining the maximum allowable temperature in the structure. Thus, assuming that this bound is kept sufficiently low allows us to treat the structure as an elastic medium whose properties remain unchanged and decoupled from the heat transfer problem. The shortcomings of such a
model are recognized, but this approach should be sufficient to gain insights and as a first step towards a more comprehensive approach.

**THE OPTIMIZATION PROBLEM**

Under the assumptions stated in the previous section the optimal layouts of structure and fire-retardant materials are controlled by two de-coupled equations of elasticity and steady-state conductive heat transfer. For simplicity of exposition we limit our attention to problems in two spatial dimensions defined over a design domain \( \Omega \) of prescribed shape.

In our formulation \( \Omega \) is decomposed into disjoint regions occupied by *structure* and fire-protective material (*insulation*). A third region, disjoint from the other two, is necessary to model the source of *heat*. We label these three regions \( \Omega_S, \Omega_I, \Omega_H \) (Figure 1) where

\[
\Omega = \Omega_S \cup \Omega_I \cup \Omega_H \quad \text{and} \quad \Omega_S \cap \Omega_I \cap \Omega_H = \emptyset
\]

and

- \( \Omega_S \) is the portion of \( \Omega \) occupied by the structure. Material here has high stiffness and high conductivity.
- \( \Omega_I \) is the portion of \( \Omega \) occupied by fire-protective material. Material here has low stiffness and low conductivity.
- \( \Omega_H \) is the portion of \( \Omega \) filled by the heat source. This models the fire.

The material in \( \Omega \) has the properties

\[
k(x) = \begin{cases} 
  k^S & \text{if } x \in \Omega_S \\
  k^I & \text{if } x \in \Omega_I \cup \Omega_H
\end{cases}
\]

and

\[
E(x) = \begin{cases} 
  E^S & \text{if } x \in \Omega_S \\
  0 & \text{if } x \in \Omega_I \cup \Omega_H
\end{cases}
\]

The properties of the *reference* materials, \( k^S, k^I \), and \( E^S \) are prescribed. \( k^S \) and \( k^I \) are matrices, respectively, the conductivity of the structure and the insulating material, where \( k^S > k^I \) and \( k^I \) is “low”. \( E^S \) is the elastic tensor of the structure which, in this discussion, is assumed to be isotropic. In addition, modeling of the heat source is introduced by defining

\[
q_h(x) = \begin{cases} 
  q & \text{if } x \in \Omega_H \\
  0 & \text{if } x \in \Omega_S \cup \Omega_I
\end{cases}
\]

\( q \) is the strength of the heat source in \( \Omega_H \).

The objective of the layout optimization problem is to find an optimal partition of \( \Omega \) into \( \Omega_S, \Omega_I, \text{and} \Omega_H \).

**Analysis**

Under the stated assumptions the simplified model of the physical problem consists of the following de-coupled equations of steady-state conductive heat transfer and elasticity:

**Heat Conduction** The temperature distribution \( T \in V^0_T(\Omega) \) satisfies

\[
a_r(T, \tilde{T}) = q(\tilde{T}) \quad \forall \tilde{T} \in V^0_T(\Omega)
\]

where \( a_r \) is the usual bi-linear form,

\[
a_r(T, \tilde{T}) = \int_{\Omega} k_{ij} \frac{\partial T}{\partial x_i} \frac{\partial \tilde{T}}{\partial x_j} d\Omega
\]

and \( q(\tilde{T}) \) is

\[
q(\tilde{T}) = \int_{\Omega} q_h \tilde{T} d\Omega
\]

The spatial distribution of material is controlled via \( k_{ij} \) in (5), the conductivity of the material, and \( q_h \) in (6), the strength of the heat source. In our problem, this heat source is the mechanism through which we model the fire. Finally, \( V^0_T(\Omega) \) is used to represent kinematic admissibility (for simplicity, only homogeneous boundary conditions are considered, Fig. 2 (a)).

**Plane Elasticity** The displacement field \( u \in V^0_u(\Omega) \) satisfies

\[
a_u(u,\bar{u}) = f(\bar{u}) \quad \forall \bar{u} \in V_u(\Omega_S)
\]

where \( a_u \) is the bi-linear form in elasticity,

\[
a_u(u,\bar{u}) = \int_{\Omega_S} E \varepsilon(u) : \varepsilon(\bar{u}) d\Omega
\]

and \( f(\bar{u}) \) is

\[
f(\bar{u}) = \int_{\Gamma} \bar{u} d\Gamma
\]
Here \( t \) is a prescribed traction applied on \( \Gamma_t \), a portion of the boundary of \( \Omega_t \). The spatial distribution of material is controlled through \( E \) in (8), the elastic tensor. Kinematic admissibility is introduced via \( V^*(\Omega_s) \) where, again, only homogeneous boundary conditions are considered (Fig. 2 (b)).

The potential impact of the temperature rise on the structure is controlled by limiting the average temperature in the structure to be below a prescribed limit. This is done through the following constraint:

\[
\left( \frac{1}{\text{meas}(\Omega_s)} \int_{\Omega_s} T_m^w \, d\Omega \right)^{1/m} \leq \overline{T} \quad \text{for integer } m \geq 1
\]

or, more simply,

\[
\int_{\Omega_t} (T_m^w - \overline{T}) \, d\Omega \leq 0 \tag{16}
\]

\( \overline{T} \) is a prescribed temperature value (here \( T > 0 \) if \( \overline{T} > 0 \)). As the control \( \rho_1 \) acts as a membership function for the fuzzy set \( \Omega_s \), constraint (16) is replaced in computations by

\[
\int_{\Omega_t} (T_m^w - \overline{T}) \rho_1 \, d\Omega \leq 0 \tag{17}
\]

The rest of the optimization problem is standard. The objective is to minimize the mean compliance of the structure. Isoperimetric constraints limit the total structural and insulation resources. This completes the formulation of the optimization problem, which is:

\[
\text{Find } \rho_1 \text{ and } \rho_2 \text{ that}
\]

\[
\text{Minimize } f(u) \tag{18}
\]

\[
\text{Subject to}
\]

\[
\int_{\Omega_t} (T_m^w - \overline{T}) \rho_1 \, d\Omega \leq 0 \tag{a}
\]

\[
\int_{\Omega_t} \rho_1 \, d\Omega \leq r_s \text{meas}(\Omega) \tag{b}
\]

\[
\int_{\Omega_t} (1 - \rho_1)(1 - \rho_2) \, d\Omega \leq r_i \text{meas}(\Omega) \tag{c}
\]

\[
0 < \rho_{\min} \leq \rho_1 \leq 1 \quad \text{and} \quad 0 \leq \rho_2 \leq 1 \tag{d}
\]

In addition, \( u \) satisfies equilibrium (7) (defined over the whole domain \( \Omega \) with \( E \) as in (11)), \( T \) satisfies (4) and \( r_s \) and \( r_i \) are prescribed percentages of the domain occupied, respectively, by structural material and insulation. For the problem to make sense one needs \( r_s + r_i < 1 \).

### Computation

The elasticity and heat transfer equations are modeled using a standard finite element method (4-node square elements are used in both cases). The controls \( \rho_1 \) and \( \rho_2 \) are discretized using element-wise constant functions. A standard adjoint variable sensitivity analysis is performed to compute gradients of the temperature constraint, which is enforced element-wise.
at the center of each element. To illustrate, with \( m=1 \) the discretized constraint (18) (a) takes the form

\[
g = \sum_e (\mathbf{v}^T \mathbf{T}^e - \tilde{T}) \rho_1^e
\]

where \( \mathbf{T}^e \) is the (4x1) vector of temperatures at the nodes in element \( e \), \( \rho_1^e \) is the value of \( \rho_1 \) at element \( e \) and \( \mathbf{v} = 1/4(1,1,1,1)^T \). The adjoint problem associated with this function is

\[
\mathbf{K} \lambda = \mathbf{V}
\]

where \( \mathbf{K} \) is the (finite element) “stiffness” matrix associated with the heat conduction problem and \( \mathbf{V} \) is obtained from the assembly

\[
\mathbf{V} = \sum_e \rho_1^e \mathbf{v}
\]

The gradient (sensitivity vector) of \( g \) is computed from

\[
\frac{dg}{d\rho_1^e} = \lambda \rho_1^e \text{ and } \frac{dg}{d\rho_2^e} = \lambda \rho_2^e
\]

where

\[
p_1^e = \frac{\partial q^e}{\partial \rho_1^e} \mathbf{v} - \frac{\partial q^e}{\partial \rho_1^e} \mathbf{K}^e \mathbf{T}^e \quad \text{and} \quad p_2^e = \frac{\partial q^e}{\partial \rho_2^e} \mathbf{v}
\]

and \( q^e = \bar{q} \rho_2^e (1 - \rho_2^e) \) is the strength of the heat source in element \( e \). The explicit dependence of the element “stiffness” matrix \( \mathbf{K}^e \) on \( \rho_1^e \) is easily available from the SIMP model equations (10). Finally, problem (18) is solved using the method of moving asymptotes of Svanberg \(^5\) and a mesh-independence filter from \(^6\) is used.

**EXAMPLES**

The following examples illustrate the effect of introducing temperature-related considerations into the problem. In all cases we use isotropic materials with

\[
k_{i1}^1 = 1 \quad k_{i2}^1 = 0.2 \quad \kappa^3 = 1 \quad \mu^3 = 0.412 \quad \rho_{\text{min}} = 0.01
\]

\( \kappa \) and \( \mu \) are, respectively, the bulk and shear moduli of the material. The design domain \( \Omega \) is rectangular, discretized using 32x20 square elements of unit side. The total amount of structural material is controlled by \( r_s = 0.35 \) in constraint (18) (b). Loads and kinematic boundary conditions are as shown in Fig. 3. In the absence of the temperature constraint, this problem corresponds to the very well known “8-bar truss” problem that was been exhaustively studied in the literature. The load \( F^0 \) is adjusted so that \( f^\ast \), the mean compliance of the optimal structure, equals 1 in the absence of temperature constraints. The layout of this structure is shown in Fig. 4. The examples are meant to illustrate how this solution is affected by temperature related considerations. (Note that issues related to temperature induced stresses are ignored here!)

![Figure 3. Boundary conditions used in Examples 1 and 2.](image)

![Figure 4. Solution to Examples without temperature constraints. \( f^\ast = 1 \).](image)

**Example 1: Effect of \( T \)**

In this set of solutions we wish to investigate how the optimal layout changes when increasingly tighter restrictions are imposed on the structure by reducing the acceptable (mean) temperature value \( \tilde{T} \). In this example the total amount of available insulation is controlled by \( r_i = 0.20 \) in (18) (c). The sub-domain of \( \Omega \) that is not occupied by either structure or insulation (i.e., \( \Omega_{tr} \)) is subjected to a heat source of strength \( \bar{q} = 1 \). This models the behavior of the system when a “fire” fills all available space. Boundary conditions are as shown in Fig. 3. The heat generated within \( \Omega_{tr} \) can only escape through the right side of the rectangle, where the temperature is fixed at \( T = 0 \).
A sequence of solutions for decreasing values of $T$ is shown in Fig. 5. The figure shows that, for high values of $T$, the optimal layout resembles the layout in Fig. 4 (no heat generated). However, as the target temperature is decreased, increasing amounts of structural material are shifted towards the “cool wall”, in an attempt to keep the structure cooler. For very low values of the target temperature, temperature considerations dominate and the compliance can become large. The insulating material is typically lumped between the structure and the source of heat, with higher concentrations in areas where the structure is forced to lie away from the cool wall.

Figure 6 shows a typical distribution of the heat source ($\Omega_S$). There we appreciate that in this model in the optimal solution the heat source can be near the structure, as long as this happens in areas near the cool wall, where the heat can be dissipated more quickly.

The introduction of the temperature constraint has the effect of reducing the stiffness of the optimal structure. For example, when the bound on the average temperature is $T=60$, the compliance of the resulting structure (Fig. 5 (e)) is increased by a factor of 2.41 from the compliance of the structure computed without a temperature constraint (Fig 4). In return, the mean temperature is decreased by 53%. The temperature distribution on the structures in Figures 4 and 5(e) is shown in Fig. 7.
Example 2: Effect of Amount of Insulation

In this example we investigate the effect of increasing the amount of available insulation. With a larger amount of insulation available \( r_i = 0.35 \) the structure is more protected and more of the structural material can be placed away from the cool wall (Fig. 8). Results with increasing amounts of available insulation follow similar trends as sequences of results for increasing the target temperature. Notice that in our formulation, increasing the size of \( \Omega_i \) but keeping the size of \( \Omega_s \) constant has the effect of reducing the amount of generated heat, as the size of \( \Omega_p \) is reduced.

\[
I_r = I_S \Omega_H \Omega_0.35\frac{1}{r} = (a) \Omega_s \text{ for } r_i = 0.20 \quad (b) \Omega_s \text{ for } r_i = 0.35
\]

\[
(c) \Omega_i \text{ for } r_i = 0.20 \quad (d) \Omega_i \text{ for } r_i = 0.35
\]

Figure 8. Optimal layout of structural material and insulation for \( T = 60 \) and different amounts of insulation.

Example 3: Effect of Boundary Conditions

Here we modify the initial assumptions by keeping the heat source outside the design domain (i.e., setting \( \rho_2 = 0 \)) and imposing a prescribed heat source along the left wall (Fig. 9). Heat strength \( q^0 \) is adjusted to that the total amount of heat is the same as in the previous examples. In this model the insulation fills whatever space is not occupied by the structure, turning the problem into a two-phase design. The solution is shown for \( T = 60 \) and \( T = 110 \) in Fig. 10. In this case the heat cannot be dissipated as efficiently, as it has to travel across the whole of \( \Omega \) to reach the cool wall. Therefore, even though more insulation is available, the average temperatures are higher. As a result, the layout of the structure for a given temperature target \( \bar{T} \) resemble solutions obtained for a distributed heat source (Example 1) but with for a lower \( \bar{T} \).

CONCLUSIONS

We have shown that it is possible to introduce consideration of measures to protect a structure from heat into the standard topology optimization problem with a relatively straightforward extension. The model used is admittedly very simple, yet it shows that the introduction of thermal criteria can significantly change the nature of the optimal layout.

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